

Some associative submanifolds of the squashed 7-sphere

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Abstract

The squashed 7-sphere S^7 is a 7-sphere with an Einstein metric given by the canonical variation and its cone $\mathbb{R}^8 - \{0\}$ has full holonomy $\text{Spin}(7)$. There is a canonical calibrating 4-form Φ on $\mathbb{R}^8 - \{0\}$. A minimal 3-submanifold in S^7 is called associative if its cone is calibrated by Φ .

In this paper, we classify two types of fundamental associative submanifolds in the squashed S^7 . One is obtained by the intersection with a 4-plane and the other is homogeneous. Then we study their infinitesimal associative deformations and explicitly show that all of them are integrable.

1 Introduction

A Riemannian 7-manifold (Y, g) is called a nearly parallel G_2 -manifold if its cone $(C(Y), \bar{g}) = (\mathbb{R}_{>0} \times Y, dr^2 + r^2g)$ has holonomy contained in $\text{Spin}(7)$. The existence of such a structure is equivalent to that of a spin structure with a real Killing spinor ([1]), which is also used in supergravity and superstring theory in physics. There is a canonical calibrating 4-form Φ on $C(Y)$. A 3-submanifold M in Y is called associative if its cone $C(M)$ is Cayley, i.e. it is calibrated by Φ .

By definition, Sasaki-Einstein manifolds, especially 3-Sasakian manifolds, admit nearly parallel G_2 -structures. Moreover, every compact 3-Sasakian 7-manifold admits a second nearly parallel G_2 -structure whose cone metric has full holonomy $\text{Spin}(7)$ ([4]). The 7-sphere S^7 with this second nearly parallel G_2 -structure is called the squashed S^7 .

Associative submanifolds in the standard S^7 were studied by Lotay [7]. In this paper, we study some fundamental associative submanifolds in the squashed S^7 and compare the properties.

First, we find some fundamental examples of associative submanifolds in the squashed S^7 . Fibers of the Hopf fibration $\pi : S^7 \rightarrow S^4$ are associative. More generally, the Hopf lifts of I'_1 -holomorphic curves in $\mathbb{C}P^3$ are also associative in the squashed S^7 (Proposition 4.9), where I'_1 is an almost complex structure on $\mathbb{C}P^3$ given by (4.2).

Next, we classify associative submanifolds obtained by the intersection with a 4-plane. Note that the automorphism group of the squashed S^7 is $\text{Sp}(1)\text{Sp}(2) = \text{Sp}(1) \times \text{Sp}(2)/\{\pm(1, 1)\}$ (Lemma 4.5).

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Theorem 1.1. *Let $V \subset \mathbb{R}^8 = \mathbb{C}^4$ be a 4-plane. Suppose that $V \cap S^7$ is associative in the squashed S^7 . Then up to the $\mathrm{Sp}(1)\mathrm{Sp}(2)$ -action, V is either*

$$V_1 = \{(z_1, z_2, 0, 0) \in \mathbb{C}^4; z_1, z_2 \in \mathbb{C}\} \quad \text{or} \quad V_2 = \{(z_1, 0, z_3, 0) \in \mathbb{C}^4; z_1, z_3 \in \mathbb{C}\}.$$

In other words, the space \mathcal{M} of 4-planes whose intersections with S^7 are associative is described as

$$\mathcal{M} = \mathrm{Sp}(1)\mathrm{Sp}(2)/K_1 \sqcup \mathrm{Sp}(1)\mathrm{Sp}(2)/K_2,$$

where $K_1 = \mathrm{Sp}(1)(\mathrm{Sp}(1) \times \mathrm{Sp}(1))$, and $K_2 = \mathrm{U}(1)\mathrm{U}(2)$.

Remark 1.2. We see that \mathcal{M} consists of two connected components, while the corresponding space in the standard S^7 is a homogeneous space $\mathrm{Spin}(7)/K$, where $K = \mathrm{SU}(2)^3/\mathbb{Z}_2$ ([5]).

Note that V_1 is a quaternionic plane in $\mathbb{C}^4 = \mathbb{H}^2$ and V_2 arises from a horizontal I_1 -curve of $\mathbb{C}P^3$ in the sense of Remark 4.10. Moreover, both $V_j \cap S^7$, where $j = 1, 2$, are totally geodesic submanifolds in the squashed S^7 . Actually, we should classify totally geodesic associative submanifolds, but it would be difficult because the squashed S^7 is neither a space of the constant curvature nor a symmetric space. It is just a homogeneous space $\mathrm{Sp}(1)\mathrm{Sp}(2)/\mathrm{Sp}(1)\mathrm{Sp}(1)$.

Next, we classify homogeneous associative submanifolds.

Theorem 1.3. *Let A be a connected associative 3-fold in the squashed $S^7 \subset \mathbb{C}^4$ which is the orbit of a closed Lie subgroup of $\mathrm{Sp}(1)\mathrm{Sp}(2)$. Then, up to the $\mathrm{Sp}(1)\mathrm{Sp}(2)$ -action, A is one of the following.*

1. $L_1 = V_1 \cap S^7$, where V_1 is given in Theorem 1.1,
2. $L_2 = V_2 \cap S^7$, where V_2 is given in Theorem 1.1,
3. $A_1 = T^3 \cdot \frac{1}{2}{}^t(1, 1, 1, i) \cong T^3$, where the T^3 -action is given by (6.1),
4. $A_2 = \mathrm{SU}(2) \cdot {}^t(1, 0, 0, 0) \cong \mathrm{SU}(2)/\mathbb{Z}_3$, where the $\mathrm{SU}(2)$ -action is given by (6.9),
5. $A_3 = \mathrm{SU}(2) \cdot {}^t(0, 0, 1, 0) \cong \mathrm{SU}(2)$, where the $\mathrm{SU}(2)$ -action is given by (6.9).

Remark 1.4. Since T^3 in (6.1) and $\mathrm{SU}(2)$ in (6.9) are contained in $\mathrm{SU}(4) \subset \mathrm{Spin}(7)$ by an appropriate change of coordinates, we obtain the similar orbits A_1, A_2 , and A_3 as in the standard S^7 case ([7]). However, since G_2 is not contained in $\mathrm{Sp}(1)\mathrm{Sp}(2)$, there are no corresponding associative orbits in the squashed S^7 to Lagrangian (totally real) submanifolds in S^6 classified by [9].

Remark 1.5. The examples A_1, A_2 , and A_3 are Hopf lifts of I_1' -holomorphic curves in $\mathbb{C}P^3$, where I_1' is an almost complex structure on $\mathbb{C}P^3$ given by (4.2). In particular, A_2 (resp. A_3) is a Hopf lift of a horizontal holomorphic curve (resp. a null-torsion I_1' -holomorphic curve defined in Definition 7.15) in $\mathbb{C}P^3$. Thus, unfortunately, we cannot find homogeneous examples which do not arise from other geometries as in the standard S^7 case ([7]). It is a further problem to find an associative submanifold which is not congruent to the fiber of $S^7 \rightarrow S^4$ or the Hopf lift of an I_1' -holomorphic curve in $\mathbb{C}P^3$ by the $\mathrm{Sp}(1)\mathrm{Sp}(2)$ -action. As far as the author is aware, such examples are not known so far.

However, by virtue of this property, we can explain their associative deformations.

Theorem 1.6. *The associative deformations of L_1, L_2 , and A_1 are trivial, i.e. all the associative deformations come from the $\mathrm{Sp}(1)\mathrm{Sp}(2)$ -action, while A_2 and A_3 have nontrivial associative deformations.*

All the associative deformations of A_2 consist of deformations of $p_1(A_2)$ as a horizontal holomorphic curve, i.e. those from the $\mathrm{PGL}(4, \mathbb{C})$ -action on \mathbb{CP}^3 via the Hopf lift, and those from actions of $j, k \in \mathrm{Sp}(1)$, where $p_1 : S^7 \rightarrow \mathbb{CP}^3$ is a projection.

All the associative deformations of A_3 consist of deformations of $p_1(A_3)$ as a null-torsion holomorphic curve, and those from actions of $j, k \in \mathrm{Sp}(1)$.

Remark 1.7. The deformations of the associative submanifolds in the standard S^7 are studied by the author ([6]). We could not explain the deformation space of the associative submanifold corresponding to A_3 , which did not arise from other known geometries. However, in the squashed S^7 case, the associative deformations of A_3 are explained by the property in Remark 1.5. We use the one-to-one correspondence between null-torsion I_1 -holomorphic curves and horizontal holomorphic curves in \mathbb{CP}^3 ([12]).

This paper is organized as follows. In Section 2, we review the fundamental facts of G_2 and $\mathrm{Spin}(7)$ geometry. In Section 3, we review the canonical variation and summarize some useful equations. In Section 4, we apply it to the 7-sphere S^7 and describe the nearly parallel G_2 -structure on the squashed S^7 explicitly. Then we give basic examples of associative submanifolds in the squashed S^7 . In Section 5, we prove Theorem 1.1 by choosing a “good” frame by $\mathrm{Sp}(1)\mathrm{Sp}(2)$ -action. In Section 6, we prove Theorem 1.3 as an analogue of [7], [9]. In Section 7, we prove Theorem 1.6 by using the representation theory as [6], [10].

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2 Preliminaries

2.1 G_2 and $\mathrm{Spin}(7)$ geometry

Definition 2.1. Define a 3-form φ_0 on \mathbb{R}^7 by

$$\varphi_0 = dx_{123} + dx_1(dx_{45} + dx_{67}) + dx_2(dx_{46} - dx_{57}) - dx_3(dx_{47} + dx_{56}),$$

where (x_1, \dots, x_7) is the standard coordinate on \mathbb{R}^7 and wedge signs are omitted. The Hodge dual of φ_0 is given by

$$*\varphi_0 = dx_{4567} + dx_{23}(dx_{67} + dx_{45}) + dx_{13}(dx_{57} - dx_{46}) - dx_{12}(dx_{56} + dx_{47}).$$

Decompose $\mathbb{R}^8 = \mathbb{R} \oplus \mathbb{R}^7$ and denote by x_0 the coordinate on \mathbb{R} . Define a self-dual 4-form Φ_0 on \mathbb{R}^8 by

$$\Phi_0 = dx_0 \wedge \varphi_0 + *\varphi_0.$$

If we identify $\mathbb{R}^8 \cong \mathbb{C}^4$ via $\mathbb{R}^8 \ni (x_0, \dots, x_7) \mapsto (x_0 + ix_1, x_2 + ix_3, x_4 + ix_5, x_6 + ix_7) =: (z_1, z_2, z_3, z_4) \in \mathbb{C}^4$, then Φ_0 is described as

$$\Phi_0 = \frac{1}{2}\omega_0 \wedge \omega_0 + \mathrm{Re}\Omega_0,$$

where $\omega_0 = \frac{i}{2} \sum_{j=1}^4 dz_{j\bar{j}}$ and $\Omega_0 = dz_{1234}$ are the standard Kähler form and the holomorphic volume form on \mathbb{C}^4 , respectively.

The stabilizers of φ_0 and Φ_0 are the exceptional Lie group G_2 and $\text{Spin}(7)$, respectively:

$$G_2 = \{g \in GL(7, \mathbb{R}); g^* \varphi_0 = \varphi_0\}, \quad \text{Spin}(7) = \{g \in GL(8, \mathbb{R}); g^* \Phi_0 = \Phi_0\}.$$

The Lie group G_2 fixes the standard metric $g_0 = \sum_{i=1}^7 (dx_i)^2$ and the orientation on \mathbb{R}^7 . They are uniquely determined by φ_0 via

$$6g_0(v_1, v_2) \text{vol}_{g_0} = i(v_1) \varphi_0 \wedge i(v_2) \varphi_0 \wedge \varphi_0, \quad (2.1)$$

where vol_{g_0} is a volume form of g_0 , $i(\cdot)$ is the interior product, and $v_i \in T(\mathbb{R}^7)$.

Similarly, $\text{Spin}(7)$ fixes the standard metric $h_0 = \sum_{i=1}^8 (dx_i)^2$ and the orientation on \mathbb{R}^8 . They are uniquely determined by Φ_0 via

$$\Phi_0^2 = 14 \text{vol}_{h_0}, \quad (i(w_2) i(w_1) \Phi_0)^2 \wedge \Phi_0 = 6 \|w_1 \wedge w_2\|_{h_0}^2 \text{vol}_{h_0}, \quad (2.2)$$

where vol_{h_0} is a volume form of h_0 , and $w_i \in T(\mathbb{R}^8)$.

Definition 2.2. Let Y be an oriented 7-manifold and φ a 3-form on Y . A 3-form φ is called a **G_2 -structure** on Y if for each $y \in Y$, there exists an oriented isomorphism between $T_y Y$ and \mathbb{R}^7 identifying φ_y with φ_0 . From (2.1), φ induces the metric g and the volume form on Y . A G_2 -structure φ is said to be **nearly parallel** if $d\varphi = 4 * \varphi$. We call a manifold with a nearly parallel G_2 -structure a **nearly parallel G_2 -manifold** for short. A G_2 -structure φ is called **torsion-free** if $d\varphi = 0, d * \varphi = 0$.

Let X be an oriented 8-manifold and Φ a 4-form on X . A 4-form Φ is called a **$\text{Spin}(7)$ -structure** on X if for each $x \in X$, there exists an oriented isomorphism between $T_x X$ and \mathbb{R}^8 identifying Φ_x with Φ_0 . From (2.2), Φ induces the metric h and the volume form on X . A $\text{Spin}(7)$ -structure Φ is called **torsion-free** if $d\Phi = 0$.

Lemma 2.3. [11] *A G_2 -structure φ is torsion-free if and only if $\text{Hol}(g) \subset G_2$. A $\text{Spin}(7)$ -structure Φ is torsion-free if and only if $\text{Hol}(h) \subset \text{Spin}(7)$.*

Lemma 2.4. *The 3-form φ is a nearly parallel G_2 -structure if and only if its Riemannian cone $C(Y) = \mathbb{R}_{>0} \times Y$ admits a torsion-free $\text{Spin}(7)$ -structure $\Phi = r^3 dr \wedge \varphi + r^4 * \varphi$ with the induced cone metric $\bar{g} = dr^2 + r^2 g$.*

Next, we give a summary of the facts about the submanifolds. Let Y be a manifold with a G_2 -structure φ and the induced metric g .

Lemma 2.5. [5] *For every oriented k -dimensional subspace $V^k \subset T_p Y$ ($\forall p \in Y, k = 3, 4$), we have $\varphi|_{V^3} \leq \text{vol}_{V^3}$, $*\varphi|_{V^4} \leq \text{vol}_{V^4}$. An oriented 3-submanifold $L^3 \subset Y$ is called **associative** if $\varphi|_{TL^3} = \text{vol}_{L^3}$. An oriented 4-submanifold L^4 is called **coassociative** if $*\varphi|_{TL^4} = \text{vol}_{L^4}$.*

Lemma 2.6. [5] *An oriented 3-submanifold L^3 is associative if and only if $*\varphi(v_1, v_2, v_3, \cdot) = 0$ for any $v_j \in TL^3$. An oriented 4-submanifold L^4 is coassociative if and only if $\varphi|_{TL^4} = 0$.*

Remark 2.7. Define the cross product $\times : TY \times TY \rightarrow TY$ by

$$g(u \times v, w) = \varphi(u, v, w)$$

for $u, v, w \in TY$. When L^3 is associative, there exists an orthonormal basis $\{e_1, e_2, e_3\}$ satisfying $e_3 = e_1 \times e_2$ at any point in L^3 .

Definition 2.8. Let X be a manifold with a $\text{Spin}(7)$ -structure Φ . Then for every oriented 4-dimensional subspace $W \subset T_x X$ ($\forall x \in X$), we have $\Phi|_W \leq \text{vol}_W$. An oriented 4-submanifold $N \subset X$ is called **Cayley** if $\Phi|_{TN} = \text{vol}_N$.

Lemma 2.9. Let (Y, φ, g) be a nearly parallel G_2 -manifold and $L \subset Y$ be an oriented 3-submanifold. By Lemma 2.4, $C(Y)$ is a manifold with a torsion-free $\text{Spin}(7)$ -structure Φ . Then $L \subset Y$ is associative if and only if $C(L) \subset C(Y)$ is Cayley.

Lemma 2.10. [7] There are no coassociative submanifolds of a nearly parallel G_2 -manifold (Y, φ, g) .

Proof. If L is a coassociative submanifold, we have $\varphi|_{TL} = 0$, which implies that $4\text{vol}_L = 4 * \varphi|_{TL} = d\varphi|_{TL} = 0$. This is a contradiction. \square

3 Canonical variation

3.1 Riemannian submersion

We give a summary of Chapter 9 of [2]. Let (M, g) and (B, h) be Riemannian manifolds and suppose that there exists a Riemannian submersion $\pi : (M, g) \rightarrow (B, h)$. Decompose the tangent bundle $TM = \mathcal{V} \oplus \mathcal{H}$, where a vertical distribution \mathcal{V} is a vector subbundle tangent to the fibers $\pi : M \rightarrow B$, and a horizontal distribution \mathcal{H} is the orthogonal complement bundle of \mathcal{V} . Denote by ∇ the Levi-Civita connection of g .

Definition 3.1. Define $(1,2)$ -tensors $A, T \in C^\infty(M, \otimes^2 T^*M \otimes TM)$ by

$$A_E F = (\nabla_{E^\top} F^\perp)^\top + (\nabla_{E^\top} F^\top)^\perp, \quad T_E F = (\nabla_{E^\perp} F^\perp)^\top + (\nabla_{E^\perp} F^\top)^\perp,$$

for $E, F \in \mathfrak{X}(M)$, where $\top : TM \rightarrow \mathcal{H}$ and $\perp : TM \rightarrow \mathcal{V}$ are projections.

Remark 3.2. The distribution \mathcal{H} is involutive if and only if $A \equiv 0$. The fibers of $\pi : M \rightarrow B$ are totally geodesic if and only if $T \equiv 0$.

In the following, we suppose that $\underline{T} \equiv 0$.

Lemma 3.3. Let X, Y be the horizontal vector fields, U, V be the vertical vector fields, and E, F be any vector fields on M . We have

$$\begin{aligned} A_U X &= 0, & A_U V &= 0, & A_X U &= (\nabla_X U)^\top, & A_X Y &= (\nabla_X Y)^\perp, \\ A_X Y &= -A_Y X, & A_X Y &= \frac{1}{2}[X, Y]^\perp, & g(A_X E, F) &= -g(E, A_X F). \end{aligned}$$

which implies that

$$\begin{aligned} \nabla_U V &= (\nabla_U V)^\perp, & \nabla_U X &= (\nabla_U X)^\top, \\ \nabla_X U &= (\nabla_X U)^\perp + A_X U, & \nabla_X Y &= A_X Y + (\nabla_X Y)^\top. \end{aligned}$$

3.2 Canonical Variation

For $s, t > 0$, define the **canonical variation** \tilde{g} of the Riemannian metric g on M by

$$\tilde{g}|_{\mathcal{V} \times \mathcal{V}} = s^2 g|_{\mathcal{V} \times \mathcal{V}}, \quad \tilde{g}|_{\mathcal{H} \times \mathcal{H}} = t^2 g|_{\mathcal{H} \times \mathcal{H}}, \quad \tilde{g}|_{\mathcal{H} \times \mathcal{V}} = 0.$$

Remark 3.4. Usually, we set $t = 1$ for simplicity. However, we introduce a parameter t to define the nearly parallel G_2 -structure. See Proposition 4.3.

Denote by $\tilde{\nabla}$ the Levi-Civita connection of \tilde{g} . Set $(1,2)$ -tensors \tilde{A} and \tilde{T} as in Definition 3.1.

Remark 3.5. The assumption $T \equiv 0$ implies that $\tilde{T} \equiv 0$ for all $s, t > 0$.

Under the canonical variation, the tensor A in Definition 3.1 and the Levi-Civita connection are changed as follows.

Lemma 3.6. *Let X, Y be the horizontal vector fields, and U, V be the vertical vector fields on M . We have*

$$\begin{aligned} \tilde{A}_X Y &= A_X Y, & \tilde{A}_X U &= \frac{s^2}{t^2} A_X U, \\ \tilde{\nabla}_X Y &= \nabla_X Y, & \tilde{\nabla}_U V &= \nabla_U V, \end{aligned}$$

$$\begin{aligned} \tilde{\nabla}_X U &= \frac{s^2}{t^2} (\nabla_X U)^\top + (\nabla_X U)^\perp, \\ \tilde{\nabla}_U X &= \frac{s^2}{t^2} (\nabla_U X)^\top + \left(1 - \frac{s^2}{t^2}\right) [U, X]^\top. \end{aligned}$$

This lemma implies the following useful equation.

Lemma 3.7. *For $E_1, E_2 \in \mathfrak{X}(M)$, we have*

$$\tilde{\nabla}_{E_1} E_2 - \nabla_{E_1} E_2 = \left(-1 + \frac{s^2}{t^2}\right) (A_{E_1} E_2^\perp + A_{E_2} E_1^\perp).$$

4 Nearly parallel G_2 -structure on the squashed S^7

The standard S^7 admits a canonical nearly parallel G_2 -structure. By the canonical variation, we obtain the second nearly parallel G_2 -structure on S^7 (Proposition 4.3). First, we review a 3-Sasakian structure on S^7 .

4.1 3-Sasakian structure on S^7

Consider the following Lie groups:

$$\begin{aligned} \mathrm{Sp}(1) &= \{a_1 + a_2 j \in \mathbb{H}; a_i \in \mathbb{C}, |a_1|^2 + |a_2|^2 = 1\}, \\ \mathrm{Sp}(2) &= \{g \in \mathrm{GL}(2, \mathbb{H}); g \text{ preserves the metric on } \mathbb{H}^2\} \\ &= \{g \in \mathrm{U}(4); {}^t g J g = J\} \\ &= \{(u, J\bar{u}, v, J\bar{v}); u, v \in \mathbb{C}^4, |u| = |v| = 1, \langle v, u \rangle_{\mathbb{C}} = \langle v, J\bar{u} \rangle_{\mathbb{C}} = 0\}, \end{aligned}$$

where $J = \begin{pmatrix} J' & 0 \\ 0 & J' \end{pmatrix}$, $J' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and $\langle \cdot, \cdot \rangle_{\mathbb{C}} : \mathbb{C}^4 \times \mathbb{C}^4 \rightarrow \mathbb{C}$ is the standard Hermitian metric on \mathbb{C}^4 .

Let $\text{Sp}(1) \times \text{Sp}(2)$ act on \mathbb{H}^2 by

$$(q, A) \cdot (q_1, q_2) = q(q_1, q_2) \overline{A},$$

where $(q, A) \in \text{Sp}(1) \times \text{Sp}(2)$, $(q_1, q_2) \in \mathbb{H}^2$. Via the identification $\mathbb{C}^4 \ni (z_1, \dots, z_4) \mapsto (z_1 + z_2 j, z_3 + z_4 j) \in \mathbb{H}^2$, the $\text{Sp}(1)$ -action on \mathbb{C}^4 is described as

$$(a_1 + a_2 j) \cdot u = a_1 u + a_2 J \overline{u}, \quad (4.1)$$

where $u \in \mathbb{C}^4$, and $\text{Sp}(2) \subset \text{U}(4)$ acts on \mathbb{C}^4 canonically. By definition, the $\text{Sp}(1)$ -action commutes with the $\text{Sp}(2)$ -action.

The actions of $i, j, k \in \text{Sp}(1)$ induce complex structures I_1, I_2, I_3 on \mathbb{C}^4 , respectively, and hence induce the 3-Sasakian structure $\{(\Phi_i, \xi_i, \eta_i, g)\}_{i=1,2,3}$ on S^7 , where g is the standard metric on S^7 , and a vector field $\xi_i \in \mathfrak{X}(S^7)$, a 1-form $\eta_i \in \Omega^1(S^7)$, and a $(1, 1)$ -tensor $\Phi_i \in C^\infty(S^7, \text{End}(TS^7))$ are defined by

$$\begin{aligned} (\xi_i)_z &= -I_i(z), \text{ where } z \in \mathbb{C}^4, \\ \eta_i &= g(\xi_i, \cdot), \\ \Phi_i &= \begin{cases} I_i & (\text{on } \text{Ker} \eta_i) \\ 0 & (\text{on } \mathbb{R} \xi_i). \end{cases} \end{aligned}$$

Note that the following conditions are satisfied:

$$\begin{aligned} \Phi_{i+2} &= \Phi_i \circ \Phi_{i+1} - \eta_{i+1} \otimes \xi_i = -\Phi_{i+1} \circ \Phi_i + \eta_i \otimes \xi_{i+1}, \\ \xi_{i+2} &= \Phi_i(\xi_{i+1}) = -\Phi_{i+1}(\xi_i), \\ \eta_{i+2} &= \eta_i \circ \Phi_{i+1} = -\eta_{i+1} \circ \Phi_i, \end{aligned}$$

where $i \in \mathbb{Z}/3$. These tensors are described explicitly as follows.

Lemma 4.1.

$$\begin{aligned} \xi_1 &= -i^t(z_1, z_2, z_3, z_4), \\ \xi_2 &= {}^t(\overline{z}_2, -\overline{z}_1, \overline{z}_4, -\overline{z}_3), \\ \xi_3 &= i^t(\overline{z}_2, -\overline{z}_1, \overline{z}_4, -\overline{z}_3), \end{aligned}$$

$$\begin{aligned} \eta_1 &= \text{Im} \left(\sum_{j=1}^4 z_j d\overline{z}_j \right), \quad \eta_2 + i\eta_3 = -z_1 dz_2 + z_2 dz_1 - z_3 dz_4 + z_4 dz_3, \\ d\eta_1 &= -i \sum_{j=1}^4 dz_j \overline{z}_j = -2g(\Phi_1(\cdot), \cdot), \quad d(\eta_2 + i\eta_3) = -2(dz_{12} + dz_{34}). \end{aligned}$$

4.2 Second nearly parallel G_2 -structure on S^7

Applying the canonical variation to a Riemannian submersion $\pi : S^7 \rightarrow S^4 = \mathbb{H}P^1$, we obtain the second nearly parallel G_2 -structure $(\tilde{\varphi}, \tilde{g})$ on S^7 . Denote

by $\omega_i = \frac{1}{2}D\eta_i = \frac{1}{2}d\eta_i((\cdot)^\top, (\cdot)^\top) \in \Omega^2(S^7)$ the covariant differentiation of $\frac{1}{2}\eta_i$, where $\top : TS^7 \rightarrow \mathcal{H}$ is a canonical projection. In other words, we have

$$\omega_1 = \frac{1}{2}d\eta_1 + \eta_{23}, \quad \omega_2 = \frac{1}{2}d\eta_2 + \eta_{31}, \quad \omega_3 = \frac{1}{2}d\eta_3 + \eta_{12}.$$

since $[\xi_i, \xi_{i+1}] = 2\xi_{i+2}$ for $i \in \mathbb{Z}/3$. On the other hand, it is well-known that $\frac{1}{2}d\eta_i = -g(\Phi_i(\cdot), \cdot)$. For example, see Section 2 of [10]. Then we deduce that

$$\omega_i = -g(\Phi_i(\cdot)^\top, (\cdot)^\top) \quad \text{for } i = 1, 2, 3.$$

Remark 4.2. Take any unit vector $X_0 \in \mathcal{H}$ and set $X_i = \Phi_i(X_0)$ for $i = 1, 2, 3$. Denote by $\{X^i\}$ the dual of $\{X_i\}$. Then we have

$$\omega_1 = -(X^{01} + X^{23}), \quad \omega_2 = -(X^{02} + X^{31}), \quad \omega_3 = -(X^{03} + X^{12}).$$

Proposition 4.3. [4] Define the Riemannian metric \tilde{g} , a 3-form $\tilde{\varphi} \in \Omega^3(S^7)$, and the 4-form $*\tilde{\varphi} \in \Omega^4(S^7)$ on S^7 by

$$\tilde{g}|_{\mathcal{V} \times \mathcal{V}} = \left(\frac{3}{5}\right)^2 g|_{\mathcal{V} \times \mathcal{V}}, \quad \tilde{g}|_{\mathcal{H} \times \mathcal{H}} = \left(\frac{3}{\sqrt{5}}\right)^2 g|_{\mathcal{H} \times \mathcal{H}}, \quad \tilde{g}|_{\mathcal{H} \times \mathcal{V}} = 0,$$

$$\begin{aligned} \tilde{\varphi} &= \frac{27}{25} \left(\frac{1}{5} \eta_{123} + \sum_{i=1}^3 \eta_i \wedge \omega_i \right), \\ *\tilde{\varphi} &= \frac{27}{25} \left(\frac{1}{2} \sum_{i=1}^3 \omega_i^2 + \frac{3}{5} (\eta_{23} \wedge \omega_1 + \eta_{31} \wedge \omega_2 + \eta_{12} \wedge \omega_3) \right). \end{aligned}$$

Then $(\tilde{\varphi}, \tilde{g})$ is a nearly parallel G_2 -structure with $\text{Hol}(\tilde{g}) = \text{Spin}(7)$ and $*\tilde{\varphi}$ is a Hodge dual of $\tilde{\varphi}$ with respect to \tilde{g} . We call $(S^7, \tilde{\varphi}, \tilde{g})$ the **squashed S^7** .

Outline of the proof. Set

$$\tilde{g}|_{\mathcal{V} \times \mathcal{V}} = s^2 g|_{\mathcal{V} \times \mathcal{V}}, \quad \tilde{g}|_{\mathcal{H} \times \mathcal{H}} = t^2 g|_{\mathcal{H} \times \mathcal{H}}, \quad \tilde{g}|_{\mathcal{H} \times \mathcal{V}} = 0,$$

$$\tilde{\varphi} = s^3 \eta_{123} + st^2 \sum_{i=1}^3 \eta_i \wedge \omega_i,$$

for $s, t > 0$. We find $s, t > 0$ satisfying $d\tilde{\varphi} = 4 * \tilde{\varphi}$. Setting $G_1 = s^3 \eta_{123}, G_2 = st^2 \sum_{i=1}^3 \eta_i \wedge \omega_i$, we have

$$\begin{aligned} *G_1 &= \frac{t^4}{6} \sum_{i=1}^3 \omega_i^2, \quad *G_2 = s^2 t^2 (\eta_{23} \wedge \omega_1 + \eta_{31} \wedge \omega_2 + \eta_{12} \wedge \omega_3), \\ d(\eta_{123}) &= \frac{2}{s^2 t^2} *G_2, \quad d\left(\sum_{i=1}^3 \eta_i \wedge \omega_i\right) = \frac{12}{t^4} *G_1 + \frac{2}{s^2 t^2} *G_2. \end{aligned}$$

Then we see that $d\tilde{\varphi} = \frac{12}{s} *G_1 + (\frac{2s}{t^2} + \frac{2}{s}) *G_2$, and hence $d\tilde{\varphi} = 4 * \tilde{\varphi}$ is equivalent to $s = 3/5, t = 3/\sqrt{5}$. The metric \tilde{g} is not Sasaki-Einstein, and hence satisfies $\text{Hol}(\tilde{g}) = \text{Spin}(7)$ by the classification of the dimensions of the spaces of real Killing spinors. \square

Remark 4.4. [4] Proposition 4.3 is valid for any compact 3-Sasakian manifolds. The metric \tilde{g} is Einstein if and only if $s = t$ or $s = t/\sqrt{5}$.

Since $\eta_1 = \text{Im}(t z d\bar{z})$, $\eta_2 + i\eta_3 = -d^t z \cdot Jz$, where $z = {}^t(z_1, z_2, z_3, z_4)$, $\text{Sp}(2)$ preserves η_j ($j = 1, 2, 3$). For $q = a_1 + a_2 j \in \text{Sp}(1)$, we have $(q^* \eta_1, q^* \eta_2, q^* \eta_3) = (\eta_1, \eta_2, \eta_3)^t M_q$, where $M_q \in \text{SO}(3)$ is described as

$$M_q = \begin{pmatrix} |a_1|^2 - |a_2|^2 & 2\text{Im}(a_1 \bar{a}_2) & 2\text{Re}(a_1 \bar{a}_2) \\ 2\text{Im}(a_1 a_2) & \text{Re}(a_1^2 + a_2^2) & \text{Im}(-a_1^2 + a_2^2) \\ -2\text{Re}(a_1 a_2) & \text{Im}(a_1^2 + a_2^2) & \text{Re}(a_1^2 - a_2^2) \end{pmatrix}.$$

Hence we see that $\text{Sp}(2)$ and $\text{Sp}(1)$ preserve $g|_{\mathcal{H} \times \mathcal{H}}, g|_{\mathcal{V} \times \mathcal{V}}, \tilde{g}$ and $\tilde{\varphi}$. In fact, we have the following.

Lemma 4.5. [4] *The automorphism group of the squashed $(S^7, \tilde{\varphi}, \tilde{g})$ is $\text{Sp}(1)\text{Sp}(2) = \text{Sp}(1) \times \text{Sp}(2)/\{\pm(1, 1)\}$.*

Remark 4.6. In this paper, we often consider the subgroup of $\text{Sp}(1)\text{Sp}(2)$. If there may be some confusion, denoting $\text{Sp}(1) = \text{Sp}(1)_L$ and $\text{Sp}(2) = \text{Sp}(2)_R$, we distinguish subgroups of $\text{Sp}(1)\text{Sp}(2) = \text{Sp}(1)_L \text{Sp}(2)_R$.

Lemma 4.7. *For any $E_1, E_2 \in \mathfrak{X}(S^7)$, we have*

$$\begin{aligned} \tilde{g}(E_1, E_2) &= -\frac{36}{25} \sum_{j=1}^3 \eta_j(E_1) \eta_j(E_2) + \frac{9}{5} g(E_1, E_2), \\ \tilde{\nabla}_{E_1} E_2 - \nabla_{E_1} E_2 &= \frac{4}{5} \Theta(E_1, E_2), \end{aligned}$$

where $\Theta \in C^\infty(S^7, \otimes^2 T^* S^7)$ is defined by

$$\Theta(E_1, E_2) = \sum_{i=1}^3 (\eta_i(E_1) \Phi_i(E_2) + \eta_i(E_2) \Phi_i(E_1)).$$

Proof. The first equation is proved easily and we omit the proof. Set $(s, t) = (3/5, 3/\sqrt{5})$ in Lemma 3.7. Since $A_X U = -\sum_{i=1}^3 \eta_i(U) \Phi_i(X)$ for a horizontal vector X and a vertical vector U , we have

$$\tilde{\nabla}_{E_1} E_2 - \nabla_{E_1} E_2 = \frac{4}{5} \sum_{i=1}^3 (\eta_i(E_1) \Phi_i(E_2^\top) + \eta_i(E_2) \Phi_i(E_1^\top)).$$

We easily see that the right hand side is equal to $\frac{4}{5} \Theta(E_1, E_2)$. \square

4.3 Associative submanifolds of the squashed S^7

By the definition of $\tilde{\varphi}$ in Proposition 4.3, we see the following.

Remark 4.8. There are no horizontal associative submanifolds, i.e. there are no associative submanifolds whose tangent spaces are contained in \mathcal{H} .

Let $\pi : S^7 \rightarrow S^4$ and $p_1 : S^7 \rightarrow \mathbb{C}P^3$ be the Hopf fibrations and $p_2 : \mathbb{C}P^3 \rightarrow S^4$ be the twistor fibration satisfying $\pi = p_2 \circ p_1$. Denote by $\underline{\mathcal{V}}$ and $\underline{\mathcal{H}}$ the distributions of $\mathbb{C}P^3$ induced by \mathcal{V} and \mathcal{H} , respectively. In other words, $\underline{\mathcal{V}}$ is

a vector subbundle of $TC\mathbb{P}^3$ tangent to the fibers p_2 , and $\underline{\mathcal{H}}$ is the orthogonal complement bundle of $\underline{\mathcal{V}}$. By an abuse of notation, denote by I_1 the standard complex structure on $\mathbb{C}P^3$ induced from the standard complex structure I_1 on \mathbb{C}^4 . Define the almost complex structure I_1' on $\mathbb{C}P^3$ by

$$I_1'|_{\underline{\mathcal{V}}} = -I_1|_{\underline{\mathcal{V}}}, \quad I_1'|_{\underline{\mathcal{H}}} = I_1|_{\underline{\mathcal{H}}}. \quad (4.2)$$

The almost complex structure I_1' is never integrable, and defines the nearly Kähler structure on $\mathbb{C}P^3$.

Proposition 4.9. *Let $\Sigma \subset \mathbb{C}P^3$ be an I_1' -holomorphic curve. Then the Hopf lift $p_1^{-1}(\Sigma) \subset S^7$ of Σ is associative in the squashed S^7 .*

Proof. Use the notation of Remark 4.2 and Proposition 4.3. Setting $\tilde{\eta}_i = (3/5)\eta_i$ and $\tilde{X}^i = (3/\sqrt{5})X^i$, we have

$$\tilde{\varphi} = \tilde{\eta}_1(\tilde{\eta}_{23} - \tilde{X}^{01} - \tilde{X}^{23}) - \tilde{\eta}_2(\tilde{X}^{02} + \tilde{X}^{31}) - \tilde{\eta}_3(\tilde{X}^{03} + \tilde{X}^{12}).$$

Then we obtain $\tilde{\eta}_{23} - \tilde{X}^{01} - \tilde{X}^{23} = -\tilde{G}(I_1'(\cdot), \cdot)$, where $\tilde{G} = \tilde{\eta}_2 \otimes \tilde{\eta}_2 + \tilde{\eta}_3 \otimes \tilde{\eta}_3 + \sum_{j=0}^3 \tilde{X}^j \otimes \tilde{X}^j$, which gives the proof. \square

Remark 4.10. Each fiber $F \cong S^2$ of p_2 is an obvious I_1' -holomorphic curve. Then the Hopf lift $p_1^{-1}(F) = \pi^{-1}(*)$ of F is associative. This is the intersection of a quaternionic plane and S^7 .

If $\Sigma \subset \mathbb{C}P^3$ is a horizontal I_1 -holomorphic curve, where we call the curve Σ horizontal if $T\Sigma \subset \underline{\mathcal{H}}|_{\Sigma}$, $\Sigma \subset \mathbb{C}P^3$ is also an I_1' -holomorphic curve. Thus the Hopf lift $p_1^{-1}(\Sigma)$ is associative. Since I_1 is the standard complex structure, we know many examples of these curves.

5 Classification of Cayley planes

In this section, we prove Theorem 1.1. Let $V^4 \subset \mathbb{R}^8$ be a 4-plane. We classify the associative submanifolds of the form $V \cap S^7$ by choosing a “good” frame of V by the $\mathrm{Sp}(1)\mathrm{Sp}(2)$ -action to consider the associative condition.

Suppose that V is spanned by e_0, \dots, e_3 ($e_i \in \mathbb{C}^4 = \mathbb{R}^8$). Since $\mathrm{Sp}(1)\mathrm{Sp}(2)$ acts transitively on S^7 , we may assume that

$$e_0 = {}^t(1, 0, 0, 0).$$

The stabilizer of $\mathrm{Sp}(1)\mathrm{Sp}(2)$ at e_0 is diffeomorphic to $\mathrm{Sp}(1)\mathrm{Sp}(1)$, which acts on S^7 as $[(p, q)] \cdot (q_1, q_2) = (pq_1\bar{p}, pq_2\bar{q})$, where $[(p, q)] \in \mathrm{Sp}(1)\mathrm{Sp}(1)$ and $(q_1, q_2) \in S^7 \subset \mathbb{H}^2 = \mathbb{R}^8$. Thus we may assume that

$$e_1 = {}^t(ci, 0, s, 0),$$

for $c, s \geq 0, c^2 + s^2 = 1$. Since $\{[(z, z)]; z \in \mathrm{U}(1)\} \subset \mathrm{Sp}(1)\mathrm{Sp}(1)$ fixes e_1 , by sweeping out the first entry, we may assume that

$$e_2 = {}^t(0, A_2, A_3 + iB_3, A_4 + iB_4),$$

for $A_j, B_k \in \mathbb{R}, A_2 \geq 0$.

Lemma 5.1. *We have*

$$\frac{5}{3}(e_1 \times e_2)_{e_0} = {}^t(5B_3si, 5A_4s + (-A_2c + 5B_4s)i, -B_3c + A_3ci, (-B_4c - A_2s) + A_4ci).$$

Thus denoting by e_3 the left-hand side, we see that $\text{span}_{\mathbb{R}}\{e_1, e_2, e_3\} \subset T_{e_0}S^7$ is associative. We deduce the condition by calculating $*\tilde{\varphi}(e_i, e_j, e_k, \cdot)_{e_l} = 0$ in the following cases:

- (1) $c > 0, A_2 > 0$,
- (2) $c > 0, A_2 = 0$,
- (3) $c = 0$.

Lemma 5.2. *In the case (1), the condition $*\tilde{\varphi}(e_0, e_2, e_3, \cdot)_{e_1} = 0$ is equivalent to*

- (i) $s = 0$,
- (ii) $s \neq 0, \quad A_3 = B_3 = 0, \quad c^2 - 3s^2 = 0, \quad \text{or}$
- (iii) $s \neq 0, \quad A_3 = B_3 = 0, \quad c(A_2^2 + 3A_4^2 + 3B_4^2) - 2sA_2B_4 = 0$.

We abbreviate the case that (1) and (ii) hold as the case (1)-(ii) in the following.

Lemma 5.3. *In the case (1)-(ii) or (1)-(iii), by normalizing e_2 , we may assume that $A_2^2 + A_4^2 + B_4^2 = 1$. Then $*\tilde{\varphi}(e_0, e_1, e_3, \cdot)_{e_2} = 0$ is equivalent to*

- (a) $A_4 = B_4 = 0$,
- (b) $A_4 = 0, \quad A_2^2 - 3B_4^2 = 0, \quad \text{or}$
- (c) $A_4 = 0, \quad (c^2 + 3s^2)A_2 - 2csB_4 = 0$.

Proof of Lemma 5.1. At e_0 , we have

$$\begin{aligned} \xi_1 &= {}^t(-i, 0, 0, 0), \\ \xi_2 &= {}^t(0, -1, 0, 0), \\ \xi_3 &= {}^t(0, -i, 0, 0). \end{aligned}$$

Setting $X_0 = {}^t(0, 0, 1, 0)$, we see $X_0 \in \mathcal{H}_{e_0}$. Then $X_i = \Phi_i(X_0)$ for $i = 1, 2, 3$ is described as

$$\begin{aligned} X_1 &= {}^t(0, 0, i, 0), \\ X_2 &= {}^t(0, 0, 0, 1), \\ X_3 &= {}^t(0, 0, 0, i), \end{aligned}$$

and we have

$$\begin{aligned} e_1 &= -c\xi_1 + sX_0, \\ e_2 &= -A_2\xi_2 + A_3X_0 + B_3X_1 + A_4X_2 + B_4X_3. \end{aligned}$$

By the definition of $\tilde{\varphi}$ in Proposition 4.3, we obtain

$$\begin{aligned}\tilde{\varphi}(e_1, e_2, \cdot)_{e_0} &= \frac{27}{125}c(A_2\eta_3 + 5A_3X^1 - 5B_3X^0 + 5A_4X^3 - 5B_4X^2) \\ &\quad + \frac{27}{25}s(-A_2X^2 - B_3\eta_1 - A_4\eta_2 - B_4\eta_3).\end{aligned}$$

Since $\tilde{g} = \frac{9}{25}\sum_{i=1}^3\eta_i + \frac{9}{5}\sum_{a=0}^3X^a$, we obtain the lemma. \square

Proof of Lemma 5.2. As in the proof of Lemma 5.1, we have at e_1

$$\begin{aligned}\xi_1 &= {}^t(c, 0, -is, 0), \\ \xi_2 &= {}^t(0, ic, 0, -s), \\ \xi_3 &= {}^t(0, -c, 0, -is).\end{aligned}$$

Setting $X_0 = {}^t(0, is, 0, c) \in \mathcal{H}_{e_1}$, $X_i = \Phi_i(X_0)$ for $i = 1, 2, 3$ is described as

$$\begin{aligned}X_1 &= {}^t(0, -s, 0, ic), \\ X_2 &= {}^t(is, 0, -c, 0), \\ X_3 &= {}^t(-s, 0, -ic, 0).\end{aligned}$$

Then by a direct computation, $*\tilde{\varphi}(e_0, e_2, e_3, \cdot)_{e_1} = 0$ is equivalent to

$$4s(c^2 - 3s^2)(cA_2^2 + 3cA_4^2 + 3cB_4^2 - 2sA_2B_4) = 0, \quad (5.1)$$

$$s \begin{pmatrix} c(-2s^2 + c^2) & -2s^3 & c(3s^2 + c^2) \\ 3sc & 3s^2 + c^2 & -2sc \end{pmatrix} \begin{pmatrix} A_3A_4 \\ A_2B_3 \\ B_3B_4 \end{pmatrix} = 0, \quad (5.2)$$

$$s \begin{pmatrix} sc & c^2 - 2s^2 & -(3s^2 + c^2) \\ c & -3s & -2s \end{pmatrix} \begin{pmatrix} A_2A_3 \\ A_3B_4 \\ B_3A_4 \end{pmatrix} = 0. \quad (5.3)$$

It is clear that $s = 0$ is a solution of (5.1), (5.2) and (5.3). We may assume that $s \neq 0$. From (5.2) and (5.3), we have

$$\begin{aligned}(A_3A_4, A_2B_3, B_3B_4) &= k(-(c^2 + 5s^2), 5sc, c^2), \\ (A_2A_3, A_3B_4, B_3A_4) &= l(5s, c, c),\end{aligned}$$

for $k, l \in \mathbb{R}$. Since $A_3A_4B_3B_4 = -k^2c^2(c^2 + 5s^2) = l^2c^2$, we obtain $k = l = 0$. The assumption $A_2 > 0$ gives $A_3 = B_3 = 0$. \square

Proof of Lemma 5.3. As in the proof of Lemma 5.1, we have at e_1

$$\begin{aligned}\xi_1 &= {}^t(0, -iA_2, 0, B_4 - iA_4), \\ \xi_2 &= {}^t(A_2, 0, A_4 - iB_4, 0), \\ \xi_3 &= {}^t(iA_2, 0, B_4 + iA_4, 0).\end{aligned}$$

Setting $X_0 = {}^t(A_4 + iB_4, 0, -A_2, 0) \in \mathcal{H}_{e_2}$, $X_i = \Phi_i(X_0)$ for $i = 1, 2, 3$ is described as

$$\begin{aligned}X_1 &= {}^t(-B_4 + iA_4, 0, -iA_2, 0), \\ X_2 &= {}^t(0, A_4 - iB_4, 0, -A_2), \\ X_3 &= {}^t(0, B_4 + iA_4, 0, -iA_2).\end{aligned}$$

Then by a direct computation, $*\tilde{\varphi}(e_0, e_1, e_3, \cdot)_{e_2} = 0$ is equivalent to

$$A_4\{cA_2(cA_2^2 - 2sA_2B_4 - 3cA_4^2 - 3cB_4^2) + 6B_4s(-3sA_2B_4 + 2cA_4^2 + 2cB_4^2)\} = 0, \quad (5.4)$$

$$(c^2 + 3s^2)A_2^3B_4 - 2csA_2^2B_4^2 + 3(3s^2 - c^2)A_2A_4^2B_4 - 3(c^2 + 3s^2)A_2B_4^3 - 6csA_4^4 + 6csB_4^4 = 0, \quad (5.5)$$

$$sA_2A_4(cA_2^2 - 2sA_2B_4 + 3cA_4^2 + 3cB_4^2) = 0. \quad (5.6)$$

Suppose that $A_4 \neq 0$ for a contradiction. Then (5.6) implies that

$$cA_2^2 - 2sA_2B_4 + 3cA_4^2 + 3cB_4^2 = 0. \quad (5.7)$$

Eliminating A_4^2 and B_4^2 from (5.4), we have $2A_2(cA_2 - 5sB_4)(cA_2 + sB_4) = 0$. However, the left hand side of (5.7) is greater than 0 when $B_4 = \frac{c}{5s}A_2$ or $-\frac{c}{s}A_2$. Thus we have $A_4 = 0$.

Then the left-hand sides of (5.4) and (5.6) vanish, and that of (5.5) is equal to $B_4(A_2^2 - 3B_4^2)\{(c^2 + 3s^2)A_2 - 2csB_4\}$, hence the proof is done. \square

Proof of Theorem 1.1. From Lemma 5.2 and 5.3, we consider the following cases:

Case (1)-(i) By the $\text{Sp}(1)$ -action, we may assume that $B_3 = A_4 = B_4 = 0$. Normalizing e_2 , we may assume $A_2^2 + A_3^2 = 1$. Then as in the proof of Lemma 5.2, $*\tilde{\varphi}(e_0, e_1, e_3, \cdot)_{e_2} = 0$ is equivalent to $A_3(A_2^2 - A_3^2) = 0$. Hence we have

$$(c, s, A_2, A_3, B_3, A_4, B_4) = (1, 0, 1, 0, 0, 0, 0), \quad (5.8)$$

$$\left(1, 0, \frac{\sqrt{3}}{2}, \pm \frac{1}{2}, 0, 0, 0\right). \quad (5.9)$$

Case (1)-(ii)-(a) By normalizing e_2 , we have $A_2 = 1$. Then we see

$$(c, s, A_2, A_3, B_3, A_4, B_4) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}, 1, 0, 0, 0, 0\right). \quad (5.10)$$

In case (1)-(ii)-(b), (1)-(ii)-(c), and (1)-(iii)-(b), we have the following solutions:

$$(c, s, A_2, A_3, B_3, A_4, B_4) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}, \frac{\sqrt{3}}{2}, 0, 0, 0, \pm \frac{1}{2}\right), \quad (5.11)$$

$$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0, 0, \frac{\sqrt{3}}{2}\right), \quad (5.12)$$

$$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, 0, 0, 0, \frac{1}{2}\right). \quad (5.13)$$

In case (1)-(iii)-(a) and (1)-(iii)-(c), we have no solutions.

The solution (5.8) corresponds to the \mathbb{H} -plane. The planes corresponding to (5.10), (5.11), (5.12), and (5.13) are congruent up to the $\text{Sp}(1)\text{Sp}(2)$ -action to that of (5.9), which is not associative at $(e_0 + e_1)/\sqrt{2}$ since $*\tilde{\varphi}((-e_0 + e_1)/\sqrt{2}, (e_2 - e_3)/\sqrt{2}, (e_2 + e_3)/\sqrt{2})_{(e_0+e_1)/\sqrt{2}} \neq 0$.

Case (2) We may assume that the first and the second entries of e_3 are zero. Hence we have $B_3s = A_4s = B_4s = 0$. If $s \neq 0$, we obtain the plane V_2 . If $s = 0$, the corresponding plane is congruent up to $\text{Sp}(2)$ -action to V_2 .

Case (3) We may assume that the first and the second entries of e_2 and e_3 are zero. However, this implies that $e_3 = 0$, which is a contradiction. \square

6 Classification of homogeneous associative submanifolds

In this section, we prove Theorem 1.3. First, we classify compact Lie subgroups of $\text{Sp}(1)\text{Sp}(2)$ which have 3-dimensional orbits. Let G be a compact connected Lie subgroup of $\text{Sp}(1)\text{Sp}(2)$. Suppose that G has a 3-dimensional orbit A . Since G acts on A as an isometry group, $\dim G \leq 3 \cdot (3+1)/2 = 6$ and $\dim G \neq 5$. (see [13], Chapter IV, Theorem 9.1). We only have to consider the Lie algebra $\mathfrak{g} \subset \mathfrak{sp}(1) \oplus \mathfrak{sp}(2)$ of G .

6.1 Case $\dim \mathfrak{g} = 3$

Suppose that $\dim \mathfrak{g} = 3$. By the classification of the compact Lie algebras, \mathfrak{g} is isomorphic to $\mathfrak{su}(2)$ or \mathfrak{t}^3 , where \mathfrak{t}^3 is a Lie algebra of the 3-torus T^3 . The case $\mathfrak{g} = \mathfrak{t}^3$ corresponds to the inclusion $T^3 \hookrightarrow \text{U}(1)\text{Sp}(2) \subset \text{U}(4)$ given by

$$(e^{i\alpha}, e^{i\beta}, e^{i\gamma}) \mapsto \text{diag}(e^{i(\alpha+\beta)}, e^{i(\alpha-\beta)}, e^{i(\alpha+\gamma)}, e^{i(\alpha-\gamma)}), \quad (6.1)$$

which is a maximal torus of $\text{Sp}(1)\text{Sp}(2)$ and induces the T^3 -action on S^7 . Define the basis $\{F_1, F_2, F_3\}$ of the Lie algebra $\mathfrak{t}^3 \cong \mathbb{R}^3$ of T^3 by

$$F_1 = (1, 0, 0), \quad F_2 = (0, 1, 0), \quad F_3 = (0, 0, 1). \quad (6.2)$$

Via the inclusion $\mathfrak{t}^3 \hookrightarrow \mathfrak{u}(1) \oplus \mathfrak{sp}(2)$, F_1, F_2, F_3 correspond to

$$\begin{pmatrix} i & & & \\ & i & & \\ & & i & \\ & & & i \end{pmatrix}, \begin{pmatrix} i & & & \\ & -i & & \\ & & 0 & \\ & & & 0 \end{pmatrix}, \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & i & \\ & & & -i \end{pmatrix}, \quad (6.3)$$

respectively.

When $\mathfrak{g} = \mathfrak{su}(2)$, we see that $\mathfrak{su}(2) = \mathfrak{sp}(1)_L$ or $\mathfrak{su}(2) \subset \mathfrak{sp}(2)_R$. Suppose that $\mathfrak{su}(2) \subset \mathfrak{sp}(2)_R$. Recall that any representation of the compact Lie group $\text{SU}(2)$ is completely reducible and the dimension of the real irreducible representation of $\text{SU}(2)$ is of the form $4k, 2l-1 (k, l \geq 1)$. Thus we have 3 types of inclusions $\mathfrak{su}(2) \hookrightarrow \mathfrak{so}(5)$ given by

$$\begin{aligned} \mathfrak{su}(2) &= \mathfrak{so}(3) \hookrightarrow \mathfrak{so}(5), \\ \mathfrak{su}(2) &\hookrightarrow \mathfrak{so}(4) \hookrightarrow \mathfrak{so}(5), \\ \mathfrak{su}(2) &\hookrightarrow \mathfrak{so}(5): \text{irreducibly.} \end{aligned}$$

The identification $\mathfrak{sp}(2) = \mathfrak{so}(5)$ induces three types of inclusions $\text{SU}(2) \hookrightarrow \text{Sp}(2)$. Hence we have the following four types of inclusions $\text{SU}(2) \hookrightarrow \text{Sp}(1)\text{Sp}(2)$.

1. $\text{SU}(2) = \text{Sp}(1)_L$ acting on S^7 by (4.1),

2. The inclusion $SU(2) \hookrightarrow Sp(2)$ given by

$$\begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \mapsto \begin{pmatrix} \bar{a} & -b & & \\ & a & -\bar{b} & \\ & \bar{b} & a & \\ & b & & \bar{a} \end{pmatrix}, \quad (6.4)$$

which induces the $SU(2)$ -action on S^7 . Define the basis $\{E_1, E_2, E_3\}$ of the Lie algebra $\mathfrak{su}(2)$ of $SU(2)$ satisfying $[E_i, E_{i+1}] = 2E_{i+2}$ for $i \in \mathbb{Z}/3$ by

$$E_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}. \quad (6.5)$$

Via this inclusion $\mathfrak{su}(2) \hookrightarrow \mathfrak{sp}(2)$, E_1, E_2, E_3 correspond to

$$\begin{pmatrix} & & 1 & \\ & & & 1 \\ -1 & & & \\ & -1 & & \end{pmatrix}, \begin{pmatrix} & & -i & \\ & & & i \\ -i & & & \\ & i & & \end{pmatrix}, \begin{pmatrix} & & -i & \\ & & & i \\ & & i & \\ & & & -i \end{pmatrix}, \quad (6.6)$$

respectively.

3. The inclusion $SU(2) \hookrightarrow Sp(2)$ given by

$$A \mapsto \begin{pmatrix} A & O_2 \\ O_2 & I_2 \end{pmatrix}. \quad (6.7)$$

Via this inclusion $\mathfrak{su}(2) \hookrightarrow \mathfrak{sp}(2)$, E_1, E_2, E_3 correspond to

$$\begin{pmatrix} & & 1 & \\ -1 & & & \\ & & & O_2 \end{pmatrix}, \begin{pmatrix} & & i & \\ i & & & \\ & & & O_2 \end{pmatrix}, \begin{pmatrix} & & i & \\ & & -i & \\ & & & O_2 \end{pmatrix}, \quad (6.8)$$

respectively.

4. The inclusion $SU(2) \hookrightarrow Sp(2)$ given by

$$\begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \mapsto \begin{pmatrix} a^3 & -\bar{b}^3 & \sqrt{3}a\bar{b}^2 & -\sqrt{3}a^2\bar{b} \\ b^3 & \bar{a}^3 & \sqrt{3}\bar{a}^2b & \sqrt{3}\bar{a}b^2 \\ \sqrt{3}ab^2 & -\sqrt{3}\bar{a}^2\bar{b} & \bar{a}(|a|^2 - 2|b|^2) & b(2|a|^2 - |b|^2) \\ \sqrt{3}a^2b & \sqrt{3}\bar{a}\bar{b}^2 & -\bar{b}(2|a|^2 - |b|^2) & a(|a|^2 - 2|b|^2) \end{pmatrix}, \quad (6.9)$$

which induces the $SU(2)$ -action on S^7 . This action is an irreducible representation of $SU(2)$ on \mathbb{C}^4 . This is the induced action of $SU(2)$ on $V_3 = \mathbb{C}^4$ from the standard action on \mathbb{C}^2 , where we use the notation of Lemma 7.6. Via $\mathfrak{su}(2) \hookrightarrow \mathfrak{sp}(2)$, E_1, E_2, E_3 correspond to

$$\begin{pmatrix} & & \sqrt{3} & \\ & & -\sqrt{3} & \\ & \sqrt{3} & & -2 \\ -\sqrt{3} & & 2 & \end{pmatrix}, \begin{pmatrix} & & \sqrt{3}i & \\ & & & 2i \\ \sqrt{3}i & & & 2i \end{pmatrix}, \begin{pmatrix} 3i & & & \\ & -3i & & \\ & & -i & \\ & & & i \end{pmatrix}, \quad (6.10)$$

respectively.

6.2 Case $\dim \mathfrak{g} = 4$

By the classification of the compact Lie algebras, \mathfrak{g} is isomorphic to $\mathfrak{su}(2) \oplus \mathbb{R}$. Since the inclusions $\mathfrak{su}(2) \hookrightarrow \mathfrak{sp}(1) \oplus \mathfrak{sp}(2)$ are classified, we have to find the 1-dimensional Lie subalgebras which commute with $\mathfrak{su}(2)$. Set

$$Z(\mathfrak{su}(2)) = \{X \in \mathfrak{sp}(1) \oplus \mathfrak{sp}(2); [X, Y] = 0 \text{ for any } Y \in \mathfrak{su}(2)\}.$$

First consider the case $\mathfrak{su}(2) = \mathfrak{sp}(1)_L$. Then we have $Z(\mathfrak{su}(2)) = \mathfrak{sp}(2)_R$. Take any 1-dimensional subspace $\mathfrak{k} \subset \mathfrak{sp}(2)_R$ and suppose that G is the Lie subgroup of $\mathrm{Sp}(1)\mathrm{Sp}(2)$ whose Lie algebra is $\mathfrak{su}(2) \oplus \mathfrak{k}$. Since the $\mathrm{Sp}(1)_L$ -action commutes with the $\mathrm{Sp}(2)_R$ -action, the G -orbit through $p \in S^7$ should be contained in $\mathrm{Sp}(1) \cdot p$ so that it is 3-dimensional. Thus this case is reduced to that of (4.1).

Next, suppose that $\mathfrak{su}(2) \subset \mathfrak{sp}(2)$ is induced from (6.4). In this case, we have $Z(\mathfrak{su}(2)) = \mathfrak{sp}(1)_L \oplus (\mathbb{R} \mathrm{diag}(i, -i, i, -i))_R$. The Lie subgroup $G \subset \mathrm{Sp}(2)$ whose Lie algebra is $(\mathfrak{su}(2) \oplus \mathbb{R} \mathrm{diag}(i, -i, i, -i))_R$ is $\mathrm{U}(2)$ whose restriction to $\mathrm{SU}(2)$ is given by (6.4). This $\mathrm{U}(2)$ action has the same orbits as the $\mathrm{SU}(2)$ -action. The new 3-dimensional orbits do not appear from $\mathfrak{sp}(1)_L$, and this case is reduced to that of (6.4).

Suppose that $\mathfrak{su}(2) \subset \mathfrak{sp}(2)$ is induced from (6.7). In this case, we have $Z(\mathfrak{su}(2)) = \mathfrak{sp}(1)_L \oplus \begin{pmatrix} O_2 & \\ & \mathfrak{su}(2) \end{pmatrix}_R$. This case is also reduced to that of (6.7) in the same way.

Suppose that $\mathfrak{su}(2) \subset \mathfrak{sp}(2)$ is induced from (6.9). In this case, we have $Z(\mathfrak{su}(2)) = \mathfrak{sp}(1)_L$. This case is also reduced to that of (6.9) in the same way.

6.3 Case $\dim \mathfrak{g} = 6$

By the classification of the compact Lie algebras, \mathfrak{g} is isomorphic to $\mathfrak{su}(2) \oplus \mathfrak{t}^3$ or $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$. When $\mathfrak{g} \cong \mathfrak{su}(2) \oplus \mathfrak{t}^3$, we have $\mathfrak{g} \cong \mathfrak{t}_L^1 \oplus (\mathfrak{su}(2) \oplus \mathfrak{t}^2)_R$. Since there are no 2-dimensional commutative Lie subalgebras of $\mathfrak{sp}(2)$ which commute with $\mathfrak{su}(2)$ by Section 6.2, this case does not occur.

When $\mathfrak{g} \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$, we have $G = \mathrm{Sp}(1)_L \cdot \mathrm{SU}(2)_R$ or $\begin{pmatrix} \mathrm{SU}(2) & \\ & \mathrm{SU}(2) \end{pmatrix}_R$, which reduces to the case above.

Thus we only have to consider the orbits of (6.1), (4.1), (6.4), (6.7), and (6.9).

6.4 T^3 -orbits

We classify associative submanifolds which are orbits of T^3 acting on S^7 as (6.1).

Proposition 6.1. *Up to the $\mathrm{Sp}(1)\mathrm{Sp}(2)$ -action, $T^3 \cdot \frac{1}{2}t(1, 1, 1, i)$ is the unique associative submanifold in the squashed S^7 which is an orbit of the T^3 -action.*

Remark 6.2. The associative orbit $A_1 = T^3 \cdot \frac{1}{2}t(1, 1, 1, i)$ is the Hopf lift of a I'_1 -holomorphic curve in \mathbb{CP}^3 , where I'_1 is defined by (4.2). We have

$$A_1 = \left\{ {}^t(z_1, z_2, z_3, z_4) \in S^7; \begin{array}{l} |z_1| = |z_2| = |z_3| = |z_4|, \\ \mathrm{Re}(z_1 z_2 \bar{z}_3 \bar{z}_4) = 0, \mathrm{Im}(z_1 z_2 \bar{z}_3 \bar{z}_4) < 0 \end{array} \right\},$$

which is a special Legendrian given in [5] via ${}^t(z_1, z_2, z_3, z_4) \mapsto {}^t(z_1, z_2, \bar{z}_3, \bar{z}_4)$. The inclusion (6.1) induces the metric $\frac{3}{5}(F^1)^2 + \frac{27}{50}(F^2)^2 + \frac{27}{50}(F^3)^2$, where $\{F^i\}$ is the dual of $\{F_i\}$.

Proof. Fix $p_0 = {}^t(z_1, z_2, z_3, z_4) \in S^7$ and set $A = T^3 \cdot p_0$. Then the tangent space $T_{p_0}A$ is spanned by the vectors F_i^* generated by F_i in (6.2):

$$\begin{aligned}(F_1^*)_{p_0} &= i{}^t(z_1, z_2, z_3, z_4) = -\xi_1, \\ (F_2^*)_{p_0} &= i{}^t(z_1, -z_2, 0, 0), \\ (F_3^*)_{p_0} &= i{}^t(0, 0, z_3, -z_4).\end{aligned}$$

By Lemma 2.6, we consider the condition $*\tilde{\varphi}(F_1^*, F_2^*, F_3^*, \cdot)|_{T_{p_0}S^7} = 0$. We easily see that $-i(F_1^*) * \tilde{\varphi} = (3^4/5^3)\text{Im}((\eta_2 - i\eta_3) \wedge d(\eta_2 + i\eta_3))$. From Lemma 4.1, we have

$$\begin{aligned}(\eta_2 + i\eta_3)(F_2^*) &= 2iz_1z_2, & (\eta_2 + i\eta_3)(F_3^*) &= 2iz_3z_4, \\ d(\eta_2 + i\eta_3)(F_2^*, \cdot) &= -2id(z_1z_2), & d(\eta_2 + i\eta_3)(F_3^*, \cdot) &= -2id(z_3z_4),\end{aligned}$$

which implies that the condition $*\tilde{\varphi}(F_1^*, F_2^*, F_3^*, \cdot)|_{T_{p_0}S^7} = 0$ is equivalent to $d(\text{Im}(z_1z_2\bar{z}_3\bar{z}_4)) = 0$. The restriction of this form to TS^7 is given by $d(\text{Im}(z_1z_2\bar{z}_3\bar{z}_4)) - d(\text{Im}(z_1z_2\bar{z}_3\bar{z}_4))(r\frac{\partial}{\partial r})\frac{dr}{r} = \text{Re}(\sum_{j=1}^4 \zeta_j dz_j)$, where $r\frac{\partial}{\partial r}$ is a position vector, $\frac{dr}{r}$ is its dual, and

$$\begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \\ \zeta_4 \end{pmatrix} = \begin{pmatrix} -iz_2\bar{z}_3\bar{z}_4 \\ -iz_1\bar{z}_3\bar{z}_4 \\ i\bar{z}_1\bar{z}_2z_4 \\ i\bar{z}_1\bar{z}_2z_3 \end{pmatrix} - 4\text{Im}(z_1z_2\bar{z}_3\bar{z}_4) \begin{pmatrix} \bar{z}_1 \\ \bar{z}_2 \\ \bar{z}_3 \\ \bar{z}_4 \end{pmatrix}.$$

Thus we see that the condition $*\tilde{\varphi}(F_1^*, F_2^*, F_3^*, \cdot)|_{T_{p_0}S^7} = 0$ is equivalent to $\zeta_j(p_0) = 0$ for $j = 1, \dots, 4$. On the other hand, setting

$$\Sigma = \{{}^t(x_1, x_2, x_3, x_4 + iy_4) \in S^7 \subset \mathbb{C}^4; x_j, y_4 \in \mathbb{R}, x_1, x_2, x_3 \geq 0\},$$

we have $S^7 = T^3 \cdot \Sigma$. Hence we may assume that $p_0 \in \Sigma$ and $x_1, x_2, x_3 \neq 0$ so that $T^3 \cdot p_0$ is 3-dimensional. Then we can solve $\zeta_j = 0$ easily to obtain

$$x_1 = x_2 = x_3 = 1/2, \quad x_4 = 0, \quad y_4 = \pm 1/2.$$

The T^3 -orbit through ${}^t(1, 1, 1, i)/2$ is mapped to that through ${}^t(1, 1, 1, -i)/2$ by $\begin{pmatrix} I_2 & 0 \\ 0 & K \end{pmatrix} \in \text{Sp}(2)$, where $K = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$, and we obtain the statement. \square

6.5 SU(2)-orbits

We consider the SU(2)-orbits of (4.1), (6.4), (6.7), or (6.9). First, we introduce a useful lemma to study associative orbits.

Lemma 6.3. ([9] Lemma 5.6.) *Let (V, ρ) be an orthogonal representation of SU(2), $\langle \cdot, \cdot \rangle$ be an SU(2)-invariant inner product on V , and $S_1 \subset V$ be the unit*

sphere. Let $M = \mathrm{SU}(2) \cdot p$ be a 3-dimensional orbit through $p \in S_1$. Define the function $\lambda_j : M \rightarrow \mathbb{R}$ for $j = 1, 2, 3$ by

$$\lambda_j = \langle (\rho_*(E_j))^*, (\rho_*(E_j))^* \rangle|_M,$$

where $\{E_j\}$ is a basis of $\mathfrak{su}(2)$ satisfying $[E_i, E_{i+1}] = 2E_{i+2}$ for $i \in \mathbb{Z}/3$ and $(\rho_*(E_j))^*$ is a vector field on V generated by $\rho_*(E_j) \in \mathfrak{gl}(V)$. Denote by $\{E^j\}$ the dual 1-form on M of $\{(\rho_*(E_j))^*|_M\}$. Then there exists $g \in \mathrm{SU}(2)$, the induced metric $\langle \cdot, \cdot \rangle|_M$ is described as

$$\langle \cdot, \cdot \rangle|_M = \sum_{j=1}^3 \lambda_j (E^j)^2, \quad (6.11)$$

at $g \cdot p \in M$. Moreover, $(M, \langle \cdot, \cdot \rangle|_M)$ is a space of constant curvature k if and only if $\lambda_1 = \lambda_2 = \lambda_3 = 1/k$.

Remark 6.4. ([9] Remark 5.4.) There exists $g' \in \mathrm{SU}(2)$ satisfying (6.11) and $\lambda_1(g') = \lambda_a(g)$, $\lambda_2(g') = \lambda_b(g)$, $\lambda_3(g') = \lambda_c(g)$, where $\{a, b, c\}$ is any permutation of $\{1, 2, 3\}$. Thus we can “permute” the λ_j .

6.5.1 $\mathrm{SU}(2)$ -orbits 1

If an $\mathrm{SU}(2)$ -action is given by (4.1), the orbit is the intersection of a quaternionic plane and S^7 , which is an obvious totally geodesic associative submanifold.

6.5.2 $\mathrm{SU}(2)$ -orbits 2

Consider the $\mathrm{SU}(2)$ -action given by (6.4). Let A be an $\mathrm{SU}(2)$ -orbit through $p_0 = {}^t(z_1, z_2, z_3, z_4)$. Then the tangent space to A at p_0 is spanned by the vectors E_i^* generated by E_i in (6.5):

$$\begin{aligned} (E_1^*)_{p_0} &= {}^t(z_3, z_4, -z_1, -z_2), \\ (E_2^*)_{p_0} &= i {}^t(-z_3, z_4, -z_1, z_2), \\ (E_3^*)_{p_0} &= i {}^t(-z_1, z_2, z_3, -z_4). \end{aligned}$$

We easily see that $g(E_i^*, E_j^*)_{p_0} = \delta_{ij}$, where g is the standard metric on S^7 . Then from Lemma 6.3, A is a constant curvature 1 submanifold of (S^7, g) . Thus A is of the form $V \cap S^7$, where $V \subset \mathbb{R}^8$ is a 4-plane. These associative submanifolds are classified by Theorem 1.1.

6.5.3 $\mathrm{SU}(2)$ -orbits 3

Consider the $\mathrm{SU}(2)$ -action given by (6.7). Let A be an $\mathrm{SU}(2)$ -orbit through $p_0 = {}^t(z_1, z_2, z_3, z_4)$. By the $\mathrm{SU}(2)$ -action, we may assume that $p_0 = {}^t(x_1, 0, z_3, z_4)$ where $x_1 > 0$, $z_3, z_4 \in \mathbb{C}$. Then the tangent space to A at p_0 is spanned by the vectors E_i^* generated by E_i in (6.5):

$$\begin{aligned} (E_1^*)_{p_0} &= {}^t(0, -x_1, 0, 0), \\ (E_2^*)_{p_0} &= {}^t(0, ix_1, 0, 0), \\ (E_3^*)_{p_0} &= {}^t(ix_1, 0, 0, 0). \end{aligned}$$

We compute

$$\begin{aligned}
(\eta_i(E_j^*)) &= \begin{pmatrix} 0 & 0 & -x_1^2 \\ -x_1^2 & 0 & 0 \\ 0 & -x_1^2 & 0 \end{pmatrix}, \\
\begin{pmatrix} d\eta_j(E_1^*, E_2^*) \\ d\eta_j(E_1^*, E_3^*) \\ d\eta_j(E_2^*, E_3^*) \end{pmatrix} &= \begin{pmatrix} x_1^2 & 0 & 0 \\ 0 & 0 & -x_1^2 \\ 0 & -x_1^2 & 0 \end{pmatrix}, \\
(i(E_i^*)d\eta_1) &= 2x_1 \begin{pmatrix} \text{Im}(dz_2) \\ \text{Re}(dz_2) \\ \text{Re}(dz_1) \end{pmatrix}, \quad (i(E_i^*)d(\eta_2 + i\eta_3)) = 2x_1 \begin{pmatrix} -dz_1 \\ idz_1 \\ -idz_2 \end{pmatrix}, \\
\sum_{i=1}^3 d\eta_i(E_1^*, E_2^*, E_3^*, \cdot) &= 12x_1^3 dx_1, \quad d(\eta_{123}) = 2x_1^5 dx_1.
\end{aligned}$$

Since $*\tilde{\varphi} = \frac{27}{25}(\frac{1}{8}\sum_{i=1}^3(d\eta_i)^2 + \frac{4}{5}d(\eta_{123}))$, we obtain $*\tilde{\varphi}(E_1^*, E_2^*, E_3^*, \cdot) = \frac{5}{54}x_1^3(15 + 16x_1^2)dx_1$. The restriction of dx_1 to TS^7 is given by

$$dx_1 - dx_1 \left(r \frac{\partial}{\partial r} \right) \frac{dr}{r} = dx_1 - x_1 (x_1 dx_1 + \text{Re}(z_3 dz_3 + z_4 dz_4)),$$

where $r \frac{\partial}{\partial r}$ is a position vector and $\frac{dr}{r}$ is its dual. This implies that $*\tilde{\varphi}(E_1, E_2, E_3, \cdot)|_{T_{p_0}S^7} = 0$ is equivalent to $x_1 = 1, z_3 = z_4 = 0$, and the resulting associative submanifold is $\{(z_1, z_2, 0, 0) \in \mathbb{C}^4; |z_1|^2 + |z_2|^2 = 1\}$.

6.5.4 SU(2)-orbits 4

For the SU(2)-action given by (6.9), we obtain the following.

Proposition 6.5. *Let A be an associative submanifold in the squashed S^7 which is an orbit of the SU(2)-action given in (6.9). Then up to the $\text{Sp}(1)\text{Sp}(2)$ -action,*

$$A = A_2 := \text{SU}(2) \cdot {}^t(1, 0, 0, 0) \quad \text{or} \quad A_3 := \text{SU}(2) \cdot {}^t(0, 0, 1, 0).$$

Remark 6.6. The associative orbit A_2 is the Hopf lift of a horizontal holomorphic curve

$$\{[a^3 : b^3 : \sqrt{3}ab^2 : \sqrt{3}a^2b] \in \mathbb{C}P^3; a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1\}$$

in $\mathbb{C}P^3$. This is a degree 3 $\mathbb{C}P^1$ in $\mathbb{C}P^3$ of the constant curvature called the Veronese curve. The associative orbit A_3 is the Hopf lift of a null-torsion I'_1 -holomorphic curve in $\mathbb{C}P^3$, which is defined in Definition 7.15. The inclusion (6.9) induces $\tilde{g}|_{A_2} = \frac{27}{25}(5(E^1)^2 + 5(E^2)^2 + 3(E^3)^2)$ and $\tilde{g}|_{A_3} = \frac{9}{25}(19(E^1)^2 + 19(E^2)^2 + (E^3)^2)$, where we use the notation of Lemma 6.3.

Remark 6.7. Set $A_2(a, b) := \text{SU}(2) \cdot {}^t(a, b, 0, 0)$ and $A_3(a, b) := \text{SU}(2) \cdot {}^t(0, 0, a, b)$ for $a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1$. Then by the action of $a+bj \in \text{Sp}(1)_L$, A_j is congruent to $A_j(a, b)$ ($j = 2, 3$). Via ${}^t(z_1, z_2, z_3, z_4) \mapsto {}^t(z_1, z_4, z_3, z_2)$, $A_2(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ is special Legendrian given by [8].

Proof of Proposition 6.5. Let A be an $\mathrm{SU}(2)$ -orbit through $p_0 = {}^t(z_1, z_2, z_3, z_4)$. Then the tangent space to A at p_0 is spanned by the vectors E_i^* generated by E_i in (6.5):

$$\begin{aligned}(E_1^*)_{p_0} &= {}^t(\sqrt{3}z_4, -\sqrt{3}z_3, \sqrt{3}z_2 - 2z_4, -\sqrt{3}z_1 + 2z_3), \\ (E_2^*)_{p_0} &= {}^t(\sqrt{3}iz_4, \sqrt{3}iz_3, \sqrt{3}iz_2 + 2iz_4, \sqrt{3}iz_1 + 2iz_3), \\ (E_3^*)_{p_0} &= {}^t(3iz_1, -3iz_2, -iz_3, iz_4).\end{aligned}$$

Since $\mathrm{SU}(2) \subset \mathrm{Sp}(2)$ -action preserves η_j , we have $L_{E_j^*}\eta_i = d\eta_i(E_j^*, \cdot) + d(\eta_i(E_j^*)) = 0$. Then by the equation $[E_j^*, E_{j+1}^*] = -2E_{j+2}^*$ for $j \in \mathbb{Z}/3$, we have

$$\begin{aligned}\sum_{i=1}^3 (d\eta_i)^2(E_1^*, E_2^*, E_3^*, \cdot) &= 2 \sum_{i,j=1}^3 d(\eta_i(E_j^*))^2, \\ d(\eta_{123})(E_1^*, E_2^*, E_3^*, \cdot) &= -d(\eta_{123}(E_1^*, E_2^*, E_3^*)). \end{aligned} \quad (6.12)$$

We compute

$$\begin{aligned}\eta_1(E_2^*) + i\eta_1(E_1^*) &= -2\sqrt{3}(\bar{z}_1z_4 + z_2\bar{z}_3) - 4z_3\bar{z}_4, \\ \eta_1(E_3^*) &= -3|z_1|^2 + 3|z_2|^2 + |z_3|^2 - |z_4|^2, \\ (\eta_2 + i\eta_3)(E_1^*) &= 2\sqrt{3}(z_1z_3 + z_2z_4) - 2(z_3^2 + z_4^2), \\ (\eta_2 + i\eta_3)(E_2^*) &= 2\sqrt{3}i(-z_1z_3 + z_2z_4) + 2i(-z_3^2 + z_4^2), \\ (\eta_2 + i\eta_3)(E_3^*) &= 6iz_1z_2 - 2iz_3z_4,\end{aligned}$$

Then we have $\sum_{i,j=1}^3 \eta_i(E_j^*)^2 = 9$ and $\sum_{i=1}^3 (d\eta_i)^2(E_1^*, E_2^*, E_3^*, \cdot) = 0$ by (6.12). Since $*\tilde{\varphi} = \frac{27}{25}(\frac{1}{8}\sum_{i=1}^3 (d\eta_i)^2 + \frac{4}{5}d(\eta_{123}))$, the condition $*\tilde{\varphi}(E_1^*, E_2^*, E_3^*, \cdot) = 0$ is equivalent to

$$d(\det M) = 0,$$

where $M = (\eta_i(E_j^*))$.

Now, we use Lemma 6.3. We may assume that $\{E_1^*, E_2^*, E_3^*\}$ are mutually orthogonal at $p_0 = {}^t(z_1, z_2, z_3, z_4)$ with respect to g . Then we have

$$z_1\bar{z}_4 - \bar{z}_2z_3 = 0, \quad \mathrm{Im}(z_1\bar{z}_3 + \bar{z}_2z_4) = 0. \quad (6.13)$$

Setting

$$\begin{aligned}\lambda_1 &= |E_1^*|^2 = 4(|z_3|^2 + |z_4|^2) - 4\sqrt{3}\mathrm{Re}(z_1\bar{z}_3 + \bar{z}_2z_4) + 3, \\ \lambda_2 &= |E_2^*|^2 = 4(|z_3|^2 + |z_4|^2) + 4\sqrt{3}\mathrm{Re}(z_1\bar{z}_3 + \bar{z}_2z_4) + 3, \\ \lambda_3 &= |E_3^*|^2 = 8(|z_1|^2 + |z_2|^2) + 1.\end{aligned}$$

We consider the following two cases as the proof of Lemma 5.7 in [7]:

- (1) all of the λ_j are distinct, (2) at least two of the λ_j are equal.

Consider the case (1). Since we can permute the λ_j by Remark 6.4, we may assume that $\lambda_3 < \lambda_1 < \lambda_2$. The inequality $\lambda_1 < \lambda_2$ implies that $\mathrm{Re}(z_1\bar{z}_3 +$

$z_2\bar{z}_4) > 0$. Thus we have $(z_1, z_2), (z_3, z_4) \neq 0$. From (6.13), there exists $\mu \in \mathbb{R}$ satisfying

$$z_3 = \mu z_1, \quad z_4 = \mu z_2. \quad (6.14)$$

Note that $\lambda_3 < \lambda_1$ is equivalent to $\mu > \sqrt{3}$. Moreover, since the $\mathrm{Sp}(1)_L$ -action commutes the $\mathrm{Sp}(2)_R$ -action and ${}^t(z_1, z_2, \mu z_1, \mu z_2)$ is mapped to $\frac{1}{\sqrt{\mu^2+1}}{}^t(1, 0, \mu, 0)$ by $(\bar{z}_1 - z_2 j)/\sqrt{|z_1|^2 + |z_2|^2} \in \mathrm{Sp}(1)_L$, we may assume that $p_0 = \frac{1}{\sqrt{\mu^2+1}}{}^t(1, 0, \mu, 0)$.

Set $v = {}^t(-\mu, 0, 1, 0) \in T_{p_0}S^7$. Then we compute

$$M_{p_0} = \frac{1}{\mu^2 + 1} \begin{pmatrix} 0 & 0 & \mu^2 - 3 \\ 2\mu(-\mu + \sqrt{3}) & 0 & 0 \\ 0 & -2\mu(\mu + \sqrt{3}) & 0 \end{pmatrix},$$

$$(v(M))_{p_0} = \frac{1}{\sqrt{\mu^2 + 1}} \begin{pmatrix} 0 & 0 & 8\mu \\ -2(\sqrt{3}\mu - 1)(\mu + \sqrt{3}) & 0 & 0 \\ 0 & 2(\sqrt{3}\mu + 1)(\mu - \sqrt{3}) & 0 \end{pmatrix},$$

where $v(M)$ is the derivative of M with respect to v . Then we have

$$\begin{aligned} d(\det M)_{p_0}(v) &= \det M_{p_0} \cdot \mathrm{tr}(v(M)M^{-1})_{p_0} \\ &= 24\mu(\mu^2 - 3)(3\mu^2 - 1)(\mu^2 + 1)^{-5/2} > 0. \end{aligned}$$

Thus we have no associative $\mathrm{SU}(2)$ -orbits in the case (1).

Next, consider the case (2). We may assume that $\lambda_1 = \lambda_2$ by Remark 6.4. Then we have $\mathrm{Re}(z_1\bar{z}_3 + z_2\bar{z}_4) = 0$, and (6.13) implies that

$$z_1\bar{z}_4 - \bar{z}_2z_3 = 0, \quad z_1\bar{z}_3 + \bar{z}_2z_4 = 0.$$

Thus,

$$\begin{aligned} z_1z_2\bar{z}_3\bar{z}_4 &= |z_2z_3|^2 = -|z_2z_4|^2 = 0, \\ \bar{z}_1\bar{z}_2z_3z_4 &= |z_1z_4|^2 = -|z_1z_3|^2 = 0. \end{aligned}$$

We deduce that either $z_1 = z_2 = 0$ or $z_3 = z_4 = 0$. Since ${}^t(z_1, z_2, 0, 0)$ (resp. ${}^t(0, 0, z_3, z_4)$) is mapped to ${}^t(1, 0, 0, 0)$ (resp. ${}^t(0, 0, 1, 0)$) by $\bar{z}_1 - z_2 j$ (resp. $\bar{z}_3 - z_4 j$) $\in \mathrm{Sp}(1)_L$, we only have to consider at $p_0 = {}^t(1, 0, 0, 0)$ or ${}^t(0, 0, 1, 0)$.

At $p_0 = {}^t(1, 0, 0, 0)$, we have

$$E_1^* = {}^t(0, 0, 0, -\sqrt{3}), \quad E_2^* = {}^t(0, 0, 0, \sqrt{3}i), \quad E_3^* = {}^t(3i, 0, 0, 0) = -3\xi_1, \quad (6.15)$$

which are also orthogonal to each other with respect to \tilde{g} and $\tilde{\varphi}(E_1^*, E_2^*, E_3^*) = -243/25 = -|E_1^*|_{\tilde{g}}|E_2^*|_{\tilde{g}}|E_3^*|_{\tilde{g}}$. At $p_0 = {}^t(0, 0, 1, 0)$, we have

$$E_1^* = {}^t(0, -\sqrt{3}, 0, 2), \quad E_2^* = {}^t(0, \sqrt{3}i, 0, 2i), \quad E_3^* = {}^t(0, 0, -i, 0) = \xi_1, \quad (6.16)$$

which are also orthogonal to each other with respect to \tilde{g} and $\tilde{\varphi}(E_1^*, E_2^*, E_3^*) = 3^3 \cdot 19/5^3 = |E_1^*|_{\tilde{g}}|E_2^*|_{\tilde{g}}|E_3^*|_{\tilde{g}}$. Thus we see that both $\mathrm{SU}(2)$ -orbits are associative. \square

7 Deformations of homogeneous associative submanifolds

We study the deformations of homogeneous associative submanifolds in the squashed S^7 . We apply the same method of [6] in the standard S^7 .

Proposition 7.1. [6] *Let (Y, φ, g) be a nearly parallel G_2 -manifold, and $A^3 \subset Y$ be an associative submanifold. Denote by ν the normal bundle of A in Y and by $\nabla^{\perp A}$ the connection on ν induced by the Levi-Civita connection ∇ of (Y, g) .*

Taking any local orthonormal frame $\{e_1, e_2, e_3\}$ of TA , define the operator $D : C^\infty(A, \nu) \rightarrow C^\infty(A, \nu)$ by

$$D\psi := \sum_{i=1}^3 e_i \times \nabla_{e_i}^{\perp A} \psi.$$

Then the vector space of all infinitesimal associative deformations of $A^3 \hookrightarrow Y$ is identified with $\{\psi \in C^\infty(A, \nu); D\psi = -\psi\}$.

Thus to compute the dimensions of the infinitesimal deformation spaces, we only have to know $\nabla^{\perp A}$ and \times . The next lemma is useful for the computation.

Lemma 7.2. *Let $\{e_1, e_2, e_3\}$ be the local oriented orthonormal frame of TA satisfying $e_3 = e_1 \times e_2$. Choose a local normal vector field V_1 with $|V_1| = 1$.*

Set $V_2 = e_1 \times V_1, V_3 = e_2 \times V_1, V_4 = -e_3 \times V_1$. Then $\{V_1, V_2, V_3, V_4\}$ is a local orthonormal frame of ν satisfying

$$\varphi = e^{123} + e^1(V^{12} + V^{34}) + e^2(V^{13} + V^{42}) - e^3(V^{14} + V^{23}),$$

where $\{e^i, V^j\}$ is a dual coframe of $\{e_i, V_j\}$. By the definition of the cross product in Remark 2.7, we have

$$(e_i \times V_j) = \begin{pmatrix} V_2 & -V_1 & V_4 & -V_3 \\ V_3 & -V_4 & -V_1 & V_2 \\ -V_4 & -V_3 & V_2 & V_1 \end{pmatrix}.$$

Lemma 7.3. [6] *For any $X, u, v \in \mathfrak{X}(A), \eta \in C^\infty(A, \nu)$, we have*

$$\nabla_X^{\perp A}(u \times \eta) = (\nabla_X^{\top A} u) \times \eta + u \times (\nabla_X^{\perp A} \eta) - (\chi(X, u, \eta))^{\perp A},$$

where $\chi(X, u, \eta) = X \times (u \times \eta) + g(X, u)\eta$ and $\top_A : TY \rightarrow TA$ and $\perp_A : TY \rightarrow \nu$ are projections.

We can compute $\nabla_{e_i}^{\perp A} V_j$ from $\nabla_{e_i}^{\top A} e_j$ and $\nabla_{e_i}^{\perp A} V_1$ by Lemma 7.3 and obtain the following. The proof is straightforward and we omit it.

Lemma 7.4. *Denote $\nabla_{e_i}^{\top A} e_j = \sum_{k=1}^3 \Gamma_{ij}^k e_k$ and $\nabla_{e_i}^{\perp A} V_1 = \sum_{j=2}^4 K_{ij} V_j$. Then we have for $i = 1, 2, 3$*

$$\begin{aligned} \nabla_{e_i}^{\perp A} V_2 &= -K_{i2} V_1 + (\Gamma_{i1}^2 - K_{i4} + \delta_{i3}) V_3 + (-\Gamma_{i1}^3 + K_{i3} + \delta_{i2}) V_4, \\ \nabla_{e_i}^{\perp A} V_3 &= -K_{i3} V_1 + (\Gamma_{i2}^1 + K_{i4} - \delta_{i3}) V_2 + (-\Gamma_{i2}^3 - K_{i2} - \delta_{i1}) V_4, \\ \nabla_{e_i}^{\perp A} V_4 &= -K_{i4} V_1 + (-\Gamma_{i3}^1 - K_{i3} - \delta_{i2}) V_3 + (-\Gamma_{i3}^2 + K_{i2} + \delta_{i1}) V_2. \end{aligned}$$

By the definition of the Levi-Civita connection, we have the following. The proof is also straightforward and we omit it.

Lemma 7.5. *Suppose that A is a Lie group G and $\{e_i\}_{i=1,2,3}$ are left invariant vector fields. Denoting $[e_i, e_j] = \sum_{k=1}^3 c_{ij}^k e_k$ ($c_{ij}^k \in \mathbb{R}$), we have*

$$\nabla_{e_i}^\top e_j = \frac{1}{2} \sum_{k=1}^3 \left(c_{ij}^k - c_{ik}^j - c_{jk}^i \right) e_k.$$

7.1 Computations on $\text{SU}(2)$

For the convenience of the computation, we summarize formulas on $\text{SU}(2)$. Define the basis $\{E_1, E_2, E_3\}$ of $\mathfrak{su}(2)$ as (6.5).

Lemma 7.6. *Let V_n be a \mathbb{C} -vector space of all complex homogeneous polynomials with two variables z_1, z_2 of degree n , where $n \geq 0$, and define the representation $\rho_n : \text{SU}(2) \rightarrow \text{GL}(V_n)$ as*

$$\left(\rho_n \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} f \right) (z_1, z_2) = f \left((z_1, z_2) \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \right).$$

Define the Hermitian inner product $\langle \cdot, \cdot \rangle$ of V_n such that

$$\left\{ v_k^{(n)} = \frac{1}{\sqrt{k!(n-k)!}} z_1^{n-k} z_2^k \right\}_{0 \leq k \leq n}$$

is a unitary basis of V_n . Denoting by $\widehat{\text{SU}(2)}$ the set of all equivalence classes of finite dimensional irreducible representations of $\text{SU}(2)$, we know that $\widehat{\text{SU}(2)} = \{(V_n, \rho_n); n \geq 0\}$. Then every \mathbb{C} -valued continuous function on $\text{SU}(2)$ is uniformly approximated by the \mathbb{C} -linear combination of the following functions:

$$\left\{ \langle \rho_n(\cdot) v_i^{(n)}, v_j^{(n)} \rangle; n \geq 0, 0 \leq i, j \leq n \right\},$$

which are mutually orthogonal with respect to the L_2 inner product.

By a direct computation, we see the following.

Lemma 7.7. *Identify $X \in \mathfrak{su}(2)$ with the left invariant differential operator on $\text{SU}(2)$. For $u = \sum_{l=0}^n C_l v_l^{(n)} \in V_n$, set*

$$u^* = \sum_{l=0}^n (-1)^{n-l} \overline{C_{n-l}} v_l^{(n)} \in V_n.$$

Then for any $n \geq 0, 0 \leq k, l \leq n, u, v \in V_n, X \in \mathfrak{su}(2)$, we have

$$\begin{aligned} X \langle \rho_n(\cdot) v, u \rangle &= \langle \rho_n(\cdot) d\rho_n(X) v, u \rangle, \\ (d\rho_n(X) v)(z_1, z_2) &= \left(\frac{\partial v}{\partial z_1}, \frac{\partial v}{\partial z_2} \right) {}^t X \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \\ \overline{\langle \rho_n(\cdot) v_k^{(n)}, u \rangle} &= (-1)^k \langle \rho_n(\cdot) v_{n-k}^{(n)}, u^* \rangle, \end{aligned}$$

$$\begin{aligned}
(-iE_1 + E_2)\langle \rho_n(\cdot)v_k^{(n)}, u \rangle &= \begin{cases} 2i\sqrt{(k+1)(n-k)}\langle \rho_n(\cdot)v_{k+1}^{(n)}, u \rangle, & (k < n) \\ 0, & (k = n) \end{cases} \\
(iE_1 + E_2)\langle \rho_n(\cdot)v_k^{(n)}, u \rangle &= \begin{cases} 2i\sqrt{k(n-k+1)}\langle \rho_n(\cdot)v_{k-1}^{(n)}, u \rangle, & (k > 0) \\ 0, & (k = 0) \end{cases} \\
iE_3\langle \rho_n(\cdot)v_k^{(n)}, u \rangle &= (-n + 2k)\langle \rho_n(\cdot)v_k^{(n)}, u \rangle.
\end{aligned}$$

Lemma 7.8. Suppose that $\{e_1, e_2, e_3\} = \{pE_1, pE_2, qE_3\}$, where $0 \neq p, q \in \mathbb{R}$, is an oriented orthonormal basis of $\mathfrak{su}(2)$ for some metric and orientation. Define the differential operator $D_{\lambda, \mu} : C^\infty(\text{SU}(2), \mathbb{R}^4) \rightarrow C^\infty(\text{SU}(2), \mathbb{R}^4)$ by

$$D_{\lambda, \mu} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = \left\{ \begin{pmatrix} 0 & -e_1 & -e_2 & e_3 \\ e_1 & 0 & e_3 & e_2 \\ e_2 & -e_3 & 0 & -e_1 \\ -e_3 & -e_2 & e_1 & 0 \end{pmatrix} + \begin{pmatrix} \lambda & & & \\ & \mu & & \\ & & \mu & \\ & & & \lambda \end{pmatrix} \right\} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}, \quad (7.1)$$

for $\lambda, \mu \in \mathbb{R}$. Setting $\Psi_1 = \psi_1 + i\psi_4, \Psi_2 = \psi_2 - i\psi_3$, $D_{\lambda, \mu}$ is described as

$$D_{\lambda, \mu} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \left\{ \begin{pmatrix} -ie_3 & -e_1 - ie_2 \\ e_1 - ie_2 & ie_3 \end{pmatrix} + \begin{pmatrix} \lambda & \\ & \mu \end{pmatrix} \right\} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}.$$

Set $\psi = {}^t(\psi_1, \psi_2, \psi_3, \psi_4)$. Then $D_{\lambda, \mu}\psi = \alpha\psi$ for $\alpha \in \mathbb{R}$ is equivalent to

$$(-ie_3 + \lambda - \alpha)\Psi_1 - (e_1 + ie_2)\Psi_2 = 0, \quad (7.2)$$

$$(e_1 - ie_2)\Psi_1 + (ie_3 + \mu - \alpha)\Psi_2 = 0. \quad (7.3)$$

These equations imply that $\Gamma_{p, q, \lambda, \mu, \alpha}\Psi_2 = 0$, where $\Gamma_{p, q, \lambda, \mu, \alpha}$ is defined by

$$\Gamma_{p, q, \lambda, \mu, \alpha} = \Delta_+ + \left(\mu - \lambda + 2q - \frac{2p^2}{q} \right) ie_3 + (-2q + \lambda - \alpha)(-\mu + \alpha), \quad (7.4)$$

where $\Delta_+ = -\sum_{i=1}^3 e_i^2$ is a Laplacian on $\text{SU}(2)$. Especially, for any $n \geq 0, 0 \leq k \leq n, u \in V_n$, we have

$$\Delta_+ \langle \rho_n(\cdot)v_k^{(n)}, u \rangle = \{(-p^2 + q^2)(n - 2k)^2 + p^2(n^2 + 2n)\} \langle \rho_n(\cdot)v_k^{(n)}, u \rangle, \quad (7.5)$$

$$\begin{aligned}
\Gamma_{p, q, \lambda, \mu, \alpha} \langle \rho_n(\cdot)v_k^{(n)}, u \rangle &= \{(-p^2 + q^2)(n - 2k)^2 \\
&\quad + p^2(n^2 + 2n) - (q(-\mu + \lambda) + 2(p^2 - q^2))(n - 2k) \\
&\quad + (-2q + \lambda - \alpha)(-\mu + \alpha)\} \langle \rho_n(\cdot)v_k^{(n)}, u \rangle. \quad (7.6)
\end{aligned}$$

Remark 7.9. In the case of $\text{SU}(2)/\Gamma$ for some finite subgroup Γ , we may consider the Γ equivariant solutions of (7.2) and (7.3).

Proof. It is straightforward to derive (7.2) and (7.3). Since $[e_1, e_2] = \frac{2p^2}{q}e_3, [e_2, e_3] = 2qe_1, [e_3, e_1] = 2qe_2$, we have $(e_1 - ie_2)ie_3 = (ie_3 + 2q)(e_1 - ie_2)$. Applying $(e_1 - ie_2)$ to (7.2), we obtain

$$(-ie_3 - 2q + \lambda - \alpha)(e_1 - ie_2)\Psi_1 + \left(-e_1^2 - e_2^2 - \frac{2p^2}{q}ie_3 \right) \Psi_2 = 0. \quad (7.7)$$

Eliminating Ψ_1 from (7.7) by (7.3) gives (7.4). From Lemma 7.7, we obtain (7.5) and (7.6). \square

7.2 The case L_1

Let $\mathrm{SU}(2) = \mathrm{Sp}(1)$ act on S^7 as (4.1). Then L_1 is the $\mathrm{SU}(2)$ -orbit through $p_0 = {}^t(1, 0, 0, 0)$. Identifying $\mathrm{SU}(2) \ni \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \mapsto a - \bar{b}j \in \mathrm{Sp}(1)$, the vector fields E_i^* generated by $E_i \in \mathfrak{su}(2)$, where $i = 1, 2, 3$, in (6.5) are described as

$$E_1^* = {}^t(1, 0, 0, 0) = -\xi_2, \quad E_2^* = {}^t(0, i, 0, 0) = -\xi_3, \quad E_3^* = {}^t(i, 0, 0, 0) = -\xi_1,$$

at p_0 , which induces the orthonormal basis $\{e_1, e_2, e_3\} = 5/3\{E_1, E_2, -E_3\}$ of $\mathfrak{su}(2)$.

Set $v_1 = \frac{\sqrt{5}}{3}{}^t(0, 0, 1, 0) \in \nu_{p_0}$, which is horizontal and $|v_1|_{\bar{g}} = 1$. Denote $X_0 = {}^t(0, 0, 1, 0)$, which is horizontal at p_0 and $X_i = \Phi_i(X_0)$ for $i = 1, 2, 3$. By the definition of $\tilde{\varphi}$ in Proposition 4.3, the vectors $v_2 = e_1 \times v_1, v_3 = e_2 \times v_1, v_4 = -e_3 \times v_1$ are described as $\{v_1, v_2, v_3, v_4\} = \frac{5}{3}\{X_0, X_2, X_3, X_1\}$. Define the vector field V_i on L_1 by $(V_i)_{g \cdot p_0} = g_* v_i$, where $g \in \mathrm{SU}(2)$, we obtain the following by Lemma 4.7 and Lemma 7.5.

$$\begin{pmatrix} \tilde{\nabla}_{e_1} V_1 \\ \tilde{\nabla}_{e_2} V_1 \\ \tilde{\nabla}_{e_3} V_1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} V_2 \\ V_3 \\ -V_4 \end{pmatrix}, \quad (\tilde{\nabla}_{e_i}^{\top_{L_1}} e_j) = \frac{5}{3} \begin{pmatrix} 0 & -e_3 & e_2 \\ e_3 & 0 & -e_1 \\ -e_2 & e_1 & 0 \end{pmatrix}.$$

This computation and Lemma 7.4 give the following.

$$(\tilde{\nabla}_{e_i} V_j) = \frac{1}{3} \begin{pmatrix} V_2 & -V_1 & V_4 & -V_3 \\ V_3 & -V_4 & -V_1 & V_2 \\ -V_4 & -V_3 & V_2 & V_1 \end{pmatrix}.$$

Then by the trivialization of ν via $\{V_1, V_2, V_3, V_4\}$, we have $D = D_{-1, -1}$, where $D_{\lambda, \mu}$ is defined in (7.1). Using the notations of Lemma 7.8, we see that Ψ_2 is constant, and hence Φ_1 is constant. Thus we obtain $\dim_{\mathbb{R}}\{\psi \in C^\infty(L_1, \nu); D\psi = -\psi\} = 4$.

Since $\dim_{\mathbb{R}} \mathrm{Sp}(1)\mathrm{Sp}(2)/\mathrm{Sp}(1)(\mathrm{Sp}(1) \times \mathrm{Sp}(1)) = 4$, $\mathrm{Sp}(1)\mathrm{Sp}(2)$ induces 4-dimensional associative deformations of L_1 and we obtain the following.

Proposition 7.10. *The associative deformations of L_1 are trivial. Its deformation space is $\mathrm{Sp}(1)\mathrm{Sp}(2)/\mathrm{Sp}(1)(\mathrm{Sp}(1) \times \mathrm{Sp}(1)) = \mathbb{H}P^1 = S^4$. The associative deformations of L_1 are the deformations of fibers of $\pi : S^7 \rightarrow S^4$ parametrized by the base space S^4 .*

7.3 The case L_2

Let $\mathrm{SU}(2)$ act on S^7 by (6.4). Then L_2 is the $\mathrm{SU}(2)$ -orbit through $p_0 = {}^t(1, 0, 0, 0)$. By (6.6), the vector fields E_i^* generated by $E_i \in \mathfrak{su}(2)$ for $i = 1, 2, 3$ in (6.5) are described as

$$E_1^* = {}^t(0, 0, -1, 0), \quad E_2^* = {}^t(0, 0, -i, 0), \quad E_3^* = {}^t(-i, 0, 0, 0) = \xi_1,$$

and satisfy $\tilde{\varphi}(E_1^*, E_2^*, E_3^*) = -27/25 < 0$ at p_0 . Then we obtain the induced oriented orthonormal basis $\{e_1, e_2, e_3\} = \{\frac{\sqrt{5}}{3}E_1, \frac{\sqrt{5}}{3}E_2, -\frac{5}{3}E_3\}$ of $\mathfrak{su}(2)$.

Set $v_1 = \frac{5}{3}{}^t(0, 1, 0, 0) = -\frac{5}{3}\xi_2 \in \nu_{p_0}$, which satisfies $|v_1|_{\bar{g}} = 1$. Denote $X_0 = {}^t(0, 0, 1, 0)$, which is horizontal at p_0 and $X_i = \Phi_i(X_0)$ for $i = 1, 2, 3$.

Since $\{e_1, e_2, e_3\} = \{-\frac{\sqrt{5}}{3}X_0, -\frac{\sqrt{5}}{3}X_1, \xi_1\}$, vectors $v_2 = e_1 \times v_1, v_3 = e_2 \times v_1, v_4 = -e_3 \times v_1$ are described as $\{v_1, v_2, v_3, v_4\} = \{-\frac{5}{3}\xi_2, \frac{\sqrt{5}}{3}X_2, -\frac{\sqrt{5}}{3}X_3, -\frac{5}{3}\xi_3\}$. Define the vector field V_i on L_2 by $(V_i)_{g \cdot p_0} = g_*v_i$ where $g \in \text{SU}(2)$. As in the case L_1 , we obtain

$$(\tilde{\nabla}_{e_i} V_j) = \frac{1}{3} \begin{pmatrix} -V_2 & V_1 & -V_4 & V_3 \\ -V_3 & V_4 & V_1 & -V_2 \\ -5V_4 & -V_3 & V_2 & 5V_1 \end{pmatrix}.$$

Then by the trivialization of ν via $\{V_1, V_2, V_3, V_4\}$, we have $D = D_{-1, -1/3}$, where $D_{\lambda, \mu}$ is defined in (7.1). Setting $(p, q, \lambda, \mu, \alpha) = (\frac{\sqrt{5}}{3}, -\frac{5}{3}, -1, \frac{1}{3}, -1)$ in (7.6), we see that

$$\Psi_2 = \langle \rho_2(\cdot)v_1^{(2)}, u \rangle$$

for $u \in V_2$. Since $\ker(e_1 - ie_2) \cap \ker(ie_3) = \mathbb{C}$, (7.2) and (7.3) imply that

$$\Psi_1 = -\frac{\sqrt{10}}{5} \langle \rho_2(\cdot)v_2^{(2)}, u \rangle + C$$

for $C \in \mathbb{C}$. Thus we obtain $\dim_{\mathbb{R}}\{\psi \in C^\infty(L_2, \nu); D\psi = -\psi\} = 8$.

Since $\dim_{\mathbb{R}} \text{Sp}(1)\text{Sp}(2)/\text{U}(1)\text{U}(2) = 8$, $\text{Sp}(1)\text{Sp}(2)$ induces 8-dimensional associative deformations of L_2 and we obtain the following.

Proposition 7.11. *The associative deformations of L_2 are trivial. Its deformation space is $\text{Sp}(1)\text{Sp}(2)/\text{U}(1)\text{U}(2)$.*

7.4 The case A_1

Let T^3 act on S^7 by (6.1). Then A_1 is the T^3 -orbit through $p_0 = \frac{1}{2}t(1, 1, 1, i)$. By (6.3), the vector fields F_i^* generated by F_i for $i = 1, 2, 3$ in (6.2) are described as

$$F_1^* = \frac{1}{2}t(i, i, i, -1) = -\xi_1, \quad F_2^* = \frac{1}{2}t(i, -i, 0, 0), \quad F_3^* = \frac{1}{2}t(0, 0, i, 1),$$

and satisfy $\tilde{\varphi}(F_1^*, F_2^*, F_3^*) = -81/250 < 0$ at p_0 . Then we obtain the induced oriented orthonormal basis $\{e_1, e_2, e_3\} = \{\frac{5}{3}F_1, \frac{5\sqrt{6}}{9}F_2, -\frac{5\sqrt{6}}{9}F_3\}$ of t^3 .

Set $v_1 = \frac{\sqrt{5}}{6}t(-1, -1, 1, i)$, which is horizontal at p_0 and $|v_1|_{\tilde{g}} = 1$. Denote $X_0 = \frac{1}{2}t(-1, -1, 1, i)$, which is horizontal at p_0 and $X_i = \Phi_i(X_0)$ for $i = 1, 2, 3$. Since

$$e_1 = -\frac{5}{3}\xi_1, \quad e_2 = \frac{5\sqrt{6}}{18}(\xi_3 + X_3), \quad e_3 = \frac{5\sqrt{6}}{18}(\xi_2 - X_2),$$

vectors $v_2 = e_1 \times v_1, v_3 = e_2 \times v_1, v_4 = -e_3 \times v_1$ are described as

$$\{v_1, v_2, v_3, v_4\} = \left\{ \frac{\sqrt{5}}{3}X_0, \frac{\sqrt{5}}{3}X_1, \frac{\sqrt{30}}{18}(-X_3 + 5\xi_3), \frac{\sqrt{30}}{18}(X_2 + 5\xi_2) \right\}.$$

Define the vector field V_i on T^3 by $(V_i)_{g \cdot p_0} = g_*v_i$, where $g \in T^3$. As in the case L_1 , we obtain

$$(\tilde{\nabla}_{e_i}^{\perp A_1} V_j) = \frac{1}{9} \begin{pmatrix} 3V_2 & -3V_1 & -12V_4 & 12V_3 \\ -2V_3 & 7V_4 & 2V_1 & -7V_2 \\ 2V_4 & 7V_3 & -7V_2 & -2V_1 \end{pmatrix}.$$

Then by the trivialization of ν via $\{V_1, V_2, V_3, V_4\}$, we have

$$D = \begin{pmatrix} 0 & -e_1 & -e_2 & e_3 \\ e_1 & 0 & e_3 & e_2 \\ e_2 & -e_3 & 0 & -e_1 \\ -e_3 & -e_2 & e_1 & 0 \end{pmatrix} + \frac{1}{9} \begin{pmatrix} 1 & & & \\ & 11 & & \\ & & 21 & \\ & & & 21 \end{pmatrix}.$$

Suppose $D\psi = -\psi$, where $\psi = {}^t(\psi_1, \psi_2, \psi_3, \psi_4)$ and $\psi_i \in C^\infty(T^3)$. Eliminating ψ_2 by $\psi_2 = -\frac{9}{20}(e_1(\psi_1) + e_3(\psi_3) + e_2(\psi_4))$, we obtain

$$\left(\frac{10}{9} + \frac{9}{20}e_1^2\right)\psi_1 + \left(\frac{9}{20}e_1e_3 - e_2\right)\psi_3 + \left(\frac{9}{20}e_1e_2 + e_3\right)\psi_4 = 0, \quad (7.8)$$

$$\left(\frac{9}{20}e_1e_3 + e_2\right)\psi_1 + \left(\frac{10}{3} + \frac{9}{20}e_3^2\right)\psi_3 + \left(\frac{9}{20}e_2e_3 - e_1\right)\psi_4 = 0, \quad (7.9)$$

$$\left(\frac{9}{20}e_1e_2 - e_3\right)\psi_1 + \left(\frac{9}{20}e_2e_3 + e_1\right)\psi_3 + \left(\frac{10}{3} + \frac{9}{20}e_2^2\right)\psi_4 = 0. \quad (7.10)$$

Define the smooth function $f_\gamma \in C^\infty(T^3, \mathbb{C})$ for $\gamma = (\gamma_1, \gamma_2, \gamma_3) \in \mathbb{Z}^3$ on $T^3 \cong (\mathbb{R}/2\pi\mathbb{Z})^3$ by $f_\gamma(\theta_1, \theta_2, \theta_3) = \exp(i \sum_{j=1}^3 \gamma_j \theta_j)$. Identifying $e_i \in \mathfrak{t}^3$ with the left invariant differential operator on T^3 , we have

$$e_1(f_\gamma) = \frac{5}{3}\gamma_1 i f_\gamma, \quad e_2(f_\gamma) = \frac{5\sqrt{6}}{9}\gamma_2 i f_\gamma, \quad e_3(f_\gamma) = -\frac{5\sqrt{6}}{9}\gamma_3 i f_\gamma.$$

By a Fourier series expansion, set

$$\psi_1 = \sum_{\gamma \in \mathbb{Z}^3} C_\gamma f_\gamma, \quad \psi_2 = \sum_{\gamma \in \mathbb{Z}^3} D_\gamma f_\gamma, \quad \psi_3 = \sum_{\gamma \in \mathbb{Z}^3} E_\gamma f_\gamma,$$

where $C_\gamma, D_\gamma, E_\gamma \in \mathbb{C}$. Then (7.8), (7.9), and (7.10) are equivalent to $M_\gamma {}^t(C_\gamma, D_\gamma, E_\gamma) = 0$, where

$$M_\gamma = \begin{pmatrix} 8 - 9\gamma_1^2 & 3\sqrt{6}\gamma_1\gamma_3 - 4\sqrt{6}\gamma_2 i & -3\sqrt{6}\gamma_1\gamma_2 - 4\sqrt{6}\gamma_3 i \\ 3\sqrt{6}\gamma_1\gamma_3 + 4\sqrt{6}\gamma_2 i & -6\gamma_3^2 + 24 & 6\gamma_2\gamma_3 - 12\gamma_1 i \\ -3\sqrt{6}\gamma_1\gamma_2 + 4\sqrt{6}\gamma_3 i & 6\gamma_2\gamma_3 + 12\gamma_1 i & -6\gamma_2^2 + 24 \end{pmatrix}.$$

To obtain a nontrivial solution ${}^t(C_\gamma, D_\gamma, E_\gamma) \neq 0$,

$$\det M_\gamma = 16 \{(9\gamma_1^2 + 6\gamma_2^2 + 6\gamma_3^2 - 22)^2 + 4(12(\gamma_2^2 + \gamma_3^2) - 49)\}$$

must vanish. We see that $\det M_\gamma = 0$ if and only if

$$(\gamma_1, \gamma_2, \gamma_3) = \pm(2, 0, 0), \pm(0, 2, 0), \pm(0, 0, 2), \pm(0, 1, 1), \pm(0, 1, -1). \quad (7.11)$$

For each γ in (7.11), we can check $\dim \ker M_\gamma = 1$. Moreover, we have $C_\gamma = \overline{C_{-\gamma}}$, $D_\gamma = \overline{D_{-\gamma}}$, and $E_\gamma = \overline{E_{-\gamma}}$ so that every ψ_j is \mathbb{R} -valued. Hence we obtain $\dim_{\mathbb{R}}\{\psi \in C^\infty(A_1, \nu); D\psi = -\psi\} = 10$.

Since $\dim_{\mathbb{R}} \mathrm{Sp}(1)\mathrm{Sp}(2)/T^3 = 10$, $\mathrm{Sp}(1)\mathrm{Sp}(2)$ induces 10-dimensional associative deformations of A_1 and we obtain the following.

Proposition 7.12. *The associative deformations of A_1 are trivial. Its deformation space is $\mathrm{Sp}(1)\mathrm{Sp}(2)/T^3$.*

7.5 The case A_2

Let $\mathrm{SU}(2)$ act on S^7 by (6.9). Then A_2 is the $\mathrm{SU}(2)$ -orbit through $p_0 = {}^t(1, 0, 0, 0)$. By (6.15), $\{e_1, e_2, e_3\} = \{\frac{\sqrt{15}}{9}E_1, \frac{\sqrt{15}}{9}E_2, -\frac{5}{9}E_3\}$ is the induced oriented orthonormal basis of $\mathfrak{su}(2)$, where $E_i \in \mathfrak{su}(2)$ for $i = 1, 2, 3$ is defined in (6.5).

Set $v_1 = \frac{5}{3}{}^t(0, 1, 0, 0) = -\frac{5}{3}\xi_2 \in \nu_{p_0}$, which satisfies $|v_1|_{\tilde{g}} = 1$. Denote $X_0 = {}^t(0, 0, 0, 1)$, which is horizontal at p_0 and $X_i = \Phi_i(X_0)$ for $i = 1, 2, 3$. Since

$$e_1 = -\frac{\sqrt{5}}{3}X_0, \quad e_2 = \frac{\sqrt{5}}{3}X_1, \quad e_3 = -\frac{5}{3}\xi_1,$$

vectors $v_2 = e_1 \times v_1, v_3 = e_2 \times v_1, v_4 = -e_3 \times v_1$ are described as

$$\{v_1, v_2, v_3, v_4\} = \left\{ -\frac{5}{3}\xi_2, \frac{\sqrt{5}}{3}X_2, \frac{\sqrt{5}}{3}X_3, \frac{5}{3}\xi_3 \right\}.$$

Define the vector field V_i in the neighborhood of p_0 of A_2 by $(V_i)_{g \cdot p_0} = g_* v_i$, where $g \in \mathrm{SU}(2)$. As in the case L_1 , we obtain

$$(\tilde{\nabla}_{e_i}^{\perp A_2} V_j) = \frac{1}{9} \begin{pmatrix} -3V_2 & 3V_1 & -3V_4 & 3V_3 \\ -3V_3 & 3V_4 & 3V_1 & -3V_2 \\ -15V_4 & 17V_3 & -17V_2 & 15V_1 \end{pmatrix}.$$

Then by the local trivialization of ν via $\{V_1, V_2, V_3, V_4\}$, we have $D = D_{-1, 23/9}$, where $D_{\lambda, \mu}$ is defined in (7.1). Setting $(p, q, \lambda, \mu, \alpha) = (\frac{\sqrt{15}}{9}, -\frac{5}{9}, -1, \frac{23}{9}, -1)$ in (7.6), we see that

$$\Psi_2 = \langle \rho_6(\cdot) v_5^{(6)}, u \rangle$$

for $u \in V_2$. Since $\ker(e_1 - ie_2) \cap \ker(ie_3) = \mathbb{C}$, (7.2) and (7.3) imply that

$$\Psi_1 = -\frac{\sqrt{10}}{5} \langle \rho_6(\cdot) v_6^{(6)}, u \rangle + C$$

for $C \in \mathbb{C}$. These solutions are \mathbb{Z}_3 -equivariant, and hence we obtain $\dim_{\mathbb{R}}\{\psi \in C^\infty(A_2, \nu); D\psi = -\psi\} = 16$.

Since $\dim_{\mathbb{R}} \mathrm{Sp}(1)\mathrm{Sp}(2)/\mathrm{U}(1)\mathrm{SU}(2) = 9$, $\mathrm{Sp}(1)\mathrm{Sp}(2)$ induces 9-dimensional associative deformations of A_2 . Thus A_2 can have at most 7-dimensional family of nontrivial associative deformations. In fact, we obtain the following.

Proposition 7.13. *All associative deformations of A_2 are induced by the $\mathrm{Sp}(1)\mathrm{Sp}(2)$ -action and by the $\mathrm{P}\mathrm{Sp}(2, \mathbb{C})$ -action on $\mathbb{C}P^3$ via the Hopf lift. In other words, all the associative deformations of A_2 are given by the following.*

- the $\mathrm{P}\mathrm{Sp}(2, \mathbb{C})$ -action on $\mathbb{C}P^3$ via the Hopf lift, which corresponds to the deformation of $p_1(A_2)$ as a horizontal holomorphic curve, where $p_1 : S^7 \rightarrow \mathbb{C}P^3$ is a projection,
- the action generated by $j, k \in \mathrm{Sp}(1)$.

Note that $\mathrm{P}\mathrm{Sp}(2, \mathbb{C})$ acts on $\mathbb{C}P^3$ as the group of biholomorphic maps which preserve the horizontal distribution [3], [10].

Proof. First description is an analogue of [10], [6] and we omit the proof. The second description follows from the next lemma. \square

Lemma 7.14. *The subgroup of $\mathrm{PSp}(2, \mathbb{C})$ which preserves $p_1(A_2)$ is isomorphic to $\mathrm{PSL}(2, \mathbb{C})$. Thus the deformation space of $p_1(A_2)$ as a holomorphic curve is $\mathrm{PSp}(2, \mathbb{C})/\mathrm{PSL}(2, \mathbb{C})$, which is 14-dimensional.*

Proof. The inclusion $\mathrm{SU}(2) \hookrightarrow \mathrm{Sp}(2)$ of (6.9) is canonically extended to $GL(2, \mathbb{C}) \hookrightarrow GL(4, \mathbb{C})$:

$$(g_{ij}) \mapsto \begin{pmatrix} g_{11}^3 & g_{12}^3 & \sqrt{3}g_{11}g_{12}^2 & \sqrt{3}g_{11}^2g_{12} \\ g_{21}^3 & g_{22}^3 & \sqrt{3}g_{21}g_{22}^2 & \sqrt{3}g_{21}^2g_{22} \\ \sqrt{3}g_{11}g_{21}^2 & \sqrt{3}g_{12}g_{22}^2 & g_{22}(g_{11}g_{22} + 2g_{12}g_{21}) & g_{21}(2g_{11}g_{22} + g_{12}g_{21}) \\ \sqrt{3}g_{11}^2g_{21} & \sqrt{3}g_{12}^2g_{22} & g_{12}(2g_{11}g_{22} + g_{12}g_{21}) & g_{11}(g_{11}g_{22} + 2g_{12}g_{21}) \end{pmatrix},$$

which is the group of biholomorphic maps which preserve $p_1(A_2)$. We can check that $GL(2, \mathbb{C}) \cap \mathrm{Sp}(2, \mathbb{C}) = \mathrm{SL}(2, \mathbb{C})$, and hence we obtain the proof. \square

7.6 The case A_3

Let $\mathrm{SU}(2)$ act on S^7 by (6.9). Then A_3 is the $\mathrm{SU}(2)$ -orbit through $p_0 = {}^t(0, 0, 1, 0)$. By (6.16), $\{e_1, e_2, e_3\} = \{\frac{5\sqrt{19}}{57}E_1, \frac{5\sqrt{19}}{57}E_2, \frac{5}{3}E_3\}$ is the induced oriented orthonormal basis of $\mathfrak{su}(2)$, where $E_i \in \mathfrak{su}(2)$ for $i = 1, 2, 3$ is defined in (6.5).

Set $v_1 = \frac{\sqrt{5}}{3}{}^t(1, 0, 0, 0) \in \nu_{p_0}$, which is horizontal at p_0 and $|v_1|_{\bar{g}} = 1$. Denote $X_0 = {}^t(1, 0, 0, 0)$, which is horizontal at p_0 and $X_i = \Phi_i(X_0)$ for $i = 1, 2, 3$. Since

$$e_1 = -\frac{5\sqrt{19}}{57}(2\xi_2 + \sqrt{3}X_2), \quad e_2 = \frac{5\sqrt{19}}{57}(-2\xi_3 + \sqrt{3}X_3), \quad e_3 = \frac{5}{3}\xi_1, \quad (7.12)$$

vectors $v_2 = e_1 \times v_1, v_3 = e_2 \times v_1, v_4 = -e_3 \times v_1$ are described as

$$\{v_1, v_2, v_3, v_4\} = \left\{ \frac{\sqrt{5}}{3}X_0, \frac{\sqrt{95}}{57}(-5\sqrt{3}\xi_2 + 2X_0), \frac{\sqrt{95}}{57}(5\sqrt{3}\xi_3 + 2X_3), \frac{\sqrt{5}}{3}X_1 \right\}.$$

Define the vector field V_i on $\mathrm{SU}(2)$ by $(V_i)_{g \cdot p_0} = g_*v_i$, where $g \in \mathrm{SU}(2)$. As in the case L_1 , we obtain

$$(\tilde{\nabla}_{e_i}^{\perp A_3} V_j) = \frac{1}{57} \begin{pmatrix} -31V_2 & 31V_1 & -31V_4 & 31V_3 \\ -31V_3 & 31V_4 & 31V_1 & -31V_2 \\ 361V_4 & -119V_3 & 119V_2 & -361V_1 \end{pmatrix}.$$

Then by the local trivialization of ν via $\{V_1, V_2, V_3, V_4\}$, we have $D = D_{141/19, -1}$, where $D_{\lambda, \mu}$ is defined in (7.1). Setting $(p, q, \lambda, \mu, \alpha) = (\frac{5\sqrt{19}}{57}, \frac{5}{3}, \frac{141}{19}, -1, -1)$ in (7.6), we see that

$$\Psi_2 = \langle \rho_6(\cdot)v_4^{(6)}, u \rangle + C$$

for $u \in V_2, C \in \mathbb{C}$. Since $\ker(ie_3 - \frac{160}{19}) = \{0\}$, (7.2) and (7.3) imply that

$$\Psi_1 = -\frac{\sqrt{190}}{10} \langle \rho_6(\cdot)v_5^{(6)}, u \rangle.$$

Hence we obtain $\dim_{\mathbb{R}}\{\psi \in C^\infty(A_3, \nu); D\psi = -\psi\} = 16$.

Since $\dim_{\mathbb{R}} \mathrm{Sp}(1)\mathrm{Sp}(2)/\mathrm{SU}(2) = 10$, $\mathrm{Sp}(1)\mathrm{Sp}(2)$ induces 10-dimensional associative deformations of A_3 . Thus A_3 can have at most 6-dimensional family of nontrivial associative deformations.

The associative deformation space of A_3 is explained by a one-to-one correspondence between null-torsion I'_1 -holomorphic curves and horizontal holomorphic curves in \mathbb{CP}^3 ([12]).

Decompose $T\mathbb{CP}^3 = \underline{\mathcal{H}} \oplus \underline{\mathcal{V}}$, where $\underline{\mathcal{V}}$ is a vector bundle tangent to the fibers of $p_2 : \mathbb{CP}^3 \rightarrow S^4$, and $\underline{\mathcal{H}}$ is its orthogonal complement bundle of $\underline{\mathcal{V}}$. Define a map $P : \underline{\mathcal{H}} - \{0\} \rightarrow \mathbb{CP}^3$ by $P(v) = [\tilde{v}]$, where $\tilde{v} \in \mathcal{H} \subset TS^7$ is a horizontal lift of v with respect to $p_1 : S^7 \rightarrow \mathbb{CP}^3$ and we identify \tilde{v} with a vector in \mathbb{C}^4 .

Let $pr_{\underline{\mathcal{H}}} : T\mathbb{CP}^3 \rightarrow \underline{\mathcal{H}}$ be a canonical projection and $\Sigma \subset \mathbb{CP}^3$ be a I'_1 -holomorphic curve with $pr_{\underline{\mathcal{H}}}|_{T\Sigma} \neq 0$. Then there exist a holomorphic line bundle $L \subset \underline{\mathcal{H}}|_{\Sigma}$ such that $pr_{\underline{\mathcal{H}}}(T\Sigma) \subset L$. If $pr_{\underline{\mathcal{H}}}$ is nowhere vanishing on Σ , $L = pr_{\underline{\mathcal{H}}}(T\Sigma)$. Denote by $L^{\perp_{\underline{\mathcal{H}}}} \subset \underline{\mathcal{H}}|_{\Sigma}$ the orthonormal complement bundle of L and set $\hat{\Sigma} = P(L^{\perp_{\underline{\mathcal{H}}}} - \{0\})$.

Definition 7.15. A non-vertical I'_1 -holomorphic curve Σ is called **null-torsion** if $\hat{\Sigma}$ is a horizontal holomorphic curve.

Proposition 7.16. [12] *There is a one-to-one correspondence between null-torsion I'_1 -holomorphic curves and horizontal holomorphic curves via $\Sigma \mapsto \hat{\Sigma}$.*

Since $p_1(A_3)$ is an image of \mathbb{CP}^1 , it is a null-torsion ([12]). We see the following.

Lemma 7.17. *By Proposition 7.16, $p_1(A_3)$ corresponds to $p_1(A_2)$.*

Proof. Since $pr_{\underline{\mathcal{H}}}$ is nowhere vanishing on $p_1(A_3)$, $L = pr_{\underline{\mathcal{H}}}(T(p_1(A_3)))$. By (7.12), $T_{p_1(p_0)}(p_1(A_3))$ is a projection of the subspace of $T_{p_0}S^7$ spanned by $-2\xi_2 - \sqrt{3}X_2$ and $-2\xi_3 + \sqrt{3}X_3$. Thus the vector bundle $\tilde{L}^{\perp_{\underline{\mathcal{H}}}}$ over A_3 whose fiber at $g \cdot p_0$, where $g \in \mathrm{SU}(2)$, is spanned by g_*X_0 and g_*X_1 satisfies $(p_1)_*(\tilde{L}^{\perp_{\underline{\mathcal{H}}}}) = L^{\perp_{\underline{\mathcal{H}}}}$, which implies that

$$\begin{aligned} \widehat{p_1(A_3)} &= [L^{\perp_{\underline{\mathcal{H}}}} - \{0\}] \\ &= \{[g^t(1, 0, 0, 0)] \in \mathbb{CP}^3; g \in \mathrm{SU}(2)\} = p_1(A_2). \end{aligned}$$

□

Remark 7.18. We easily see that $\widehat{p_1(A_1)} = p_1(A_1)$, and hence $p_1(A_1)$ is not null-torsion.

Since the deformation space of $p_1(A_2)$ as a horizontal holomorphic curve is 14-dimensional by Proposition 7.13, we obtain the following result.

Proposition 7.19. *All the associative deformations of A_3 are given by the following.*

- the Hopf lift of null-torsion I'_1 -holomorphic curves, which correspond to horizontal holomorphic curves obtained by deforming $p_1(A_2)$ by the $\mathrm{PSp}(2, \mathbb{C})$ -action on \mathbb{CP}^3 by Proposition 7.16,
- the action generated by $j, k \in \mathrm{Sp}(1)$.

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